Challenge Problem

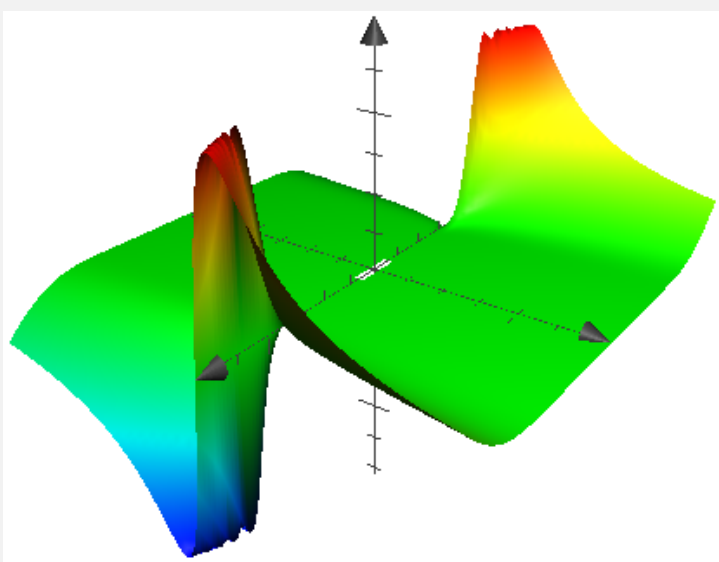
Problem (*Hmm.*)

Define $f : \mathbb{R}^2 \to \mathbb{R}$ *by*

$$\varphi(x,y) = \begin{cases} \dfrac{e^{-1/x^2} y}{e^{-2/x^2} + y^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

1. *Let* C *be an arbitrary curve* $y = c\,x^{m/n}$ *for* $m, n \in \mathbb{N}$ *with* n*: odd. Find*
$$\lim_{x\to 0} \varphi(x, c\,x^{m/n})$$
2. *Define the sequence* $\vec{a}_n = \left(\frac{1}{n}, e^{-n^2}\right)$*. Find*
$$\lim_{n\to\infty} \vec{a_n} \qquad \text{and} \qquad \lim_{n\to\infty} \varphi(\vec{a}_n)$$
3. *Is* φ *continuous at* $\vec{0}$*?*

The Challenge Problem Plot Thickens



$$\varphi(x,y) = \begin{cases} \dfrac{e^{-1/x^2} y}{e^{-2/x^2} + y^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$$

Partial Derivatives

Definition (Partial Derivatives)

Let D be an open set in $\mathbb{R}^2$, $(a,b) \in D$, and $f : D \to \mathbb{R}$. Then

$$\frac{\partial f}{\partial x}(a,b) = \lim_{h\to 0} \frac{f(a+h,b) - f(a,b)}{h},$$
$$\frac{\partial f}{\partial y}(a,b) = \lim_{h\to 0} \frac{f(a,b+h) - f(a,b)}{h}$$

when the limits are finite.

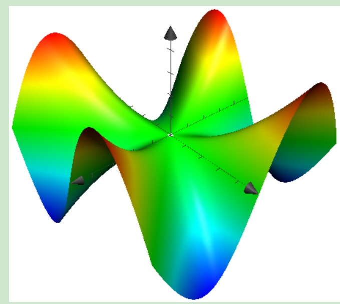
Example (*Woof!)*

Let $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$ and $f(\vec{0}) = 0$. Then

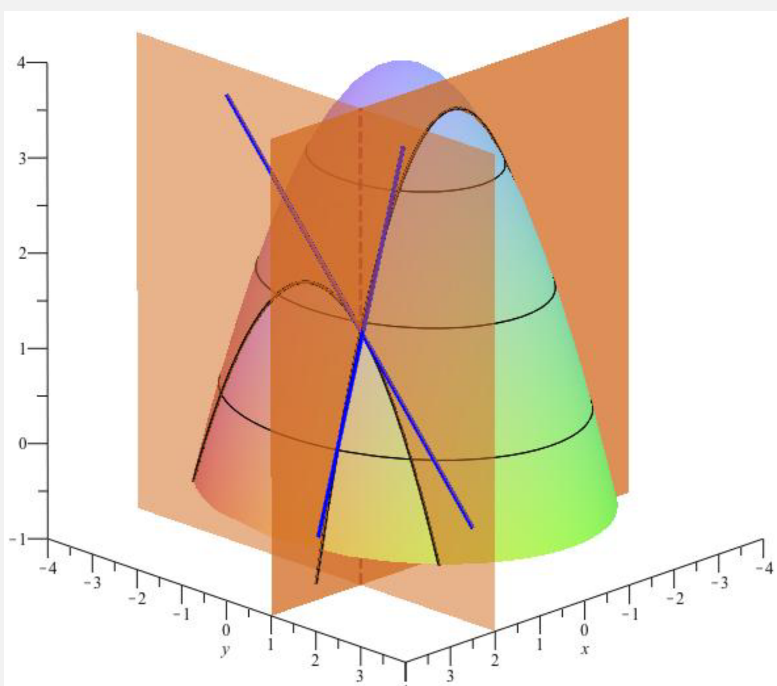
$$f_x(0,0) = \lim_{h\to 0} \frac{f(h,0)-0}{h} = 0$$

and

$$f_y(0,0) = \lim_{h\to 0} \frac{f(0,h)-0}{h} = 0$$



Picture Time



$$f(x,y) = 4 - \tfrac{1}{2}x^2 - \tfrac{1}{3}y^2 \quad \text{and} \quad \frac{\partial f}{\partial y}(2,1) \;\&\; \frac{\partial f}{\partial x}(2,1)$$

▸ Differentiation

More Partial Derivatives

Examples

1. $h(x, y) = x^2/\sqrt{y}$. Then
$$h_x(x, y) = 2x\, y^{-1/2}$$
$$h_y(x, y) = -\tfrac{1}{2} x^2 y^{-3/2}$$

2. $g(x, y) = -\cos(x + y^2)$. Then
$$g_x(x, y) = \sin(x + y^2)$$
$$g_y(x, y) = 2y \sin(x + y^2)$$

3. $f(x, y) = x^2 \sin(y) - x e^{-xy}$. Then
$$f_x(x, y) = 2x \sin(y) + (xy - 1)e^{-xy}$$
$$f_y(x, y) = x^2 \left(\cos(y) + e^{-xy}\right)$$

Deeper Partial Derivatives

Theorem (Clairaut's[3] Theorem (1743))

Let $D \subset \mathbb{R}^2$ be open and $f : D \to \mathbb{R}$. If $\frac{\partial^2 f}{\partial x \partial y}$ and $\frac{\partial^2 f}{\partial y \partial x}$ are continuous on D, then $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$ on D.

Proof.

Let $(a, b) \in D$. Set
$$g(h, k) = f(a + h, b + k) - f(a, b + k) - f(a + h, b) + f(a, b)$$
$$p(x, y) = f(x + h, y) - f(x, y) = \Delta_x f$$
$$q(x, y) = f(x, y + k) - f(x, y) = \Delta_y f$$

Then
$$g(h, k) = p(a, b + k) - p(a, b) = \Delta_y p = \Delta_y \Delta_x f$$
$$g(h, k) = q(a + h, b) - q(a, b) = \Delta_x q = \Delta_x \Delta_y f$$

[3] *Presented his first paper at age 13; only one of his 19 siblings to reach adulthood.*

Deeper Partial Derivatives, II

Proof (cont).

Apply the MVT to $\Delta_y p$ and $\Delta_x q$ above to have (for some $\theta_j \in (0,1)$)

$$g(h,k) = k\,p_y(a, b+\theta_1 k) = k \cdot [f_y(a+h, b+\theta_1 k) - f_y(a, b+\theta_1 k)]$$

$$g(h,k) = h\,q_x(a+\theta_2 h, b)) = h \cdot [f_x(a+\theta_2 h, b+k) - f_x(a+\theta_2 h, b)]$$

Apply the MVT to $\Delta_x f_y$ and $\Delta_y f_x$ above to have (for some $\theta_k \in (0,1)$).

$$g(h,k) = hk\, f_{yx}(a+\theta_3 h, b+\theta_1 k)$$
$$g(h,k) = kh\, f_{xy}(a+\theta_2 h, b+\theta_4 k)$$

Whence

$$f_{yx}(a+\theta_3 h, b+\theta_1 k) = f_{xy}(a+\theta_2 h, b+\theta_4 k)$$

Let $h, k \to 0$. Since f_{xy} and f_{yx} are continuous, then

$$f_{yx}(a,b) = f_{xy}(a,b)$$

□

Deeper Samples

Examples

1. $g(x,y) = -\cos(x+y^2)$. Then

$$g_x(x,y) = \sin(x+y^2) \implies g_{xy}(x,y) = 2y\cos(x+y^2)$$

$$g_y(x,y) = 2y\sin(x+y^2) \implies g_{yx}(x,y) = 2y\cos(x+y^2)$$

2. $f(x,y) = \dfrac{xy(x^2-y^2)}{x^2+y^2}$. Then (Maple)

$$f_y(x,0) = \begin{cases} x & x \neq 0 \\ 0 & x = 0 \end{cases}$$

$$f_x(0,y) = \begin{cases} -y & y \neq 0 \\ 0 & y = 0 \end{cases}$$

Whence $f_{xy}(0,0) = -1$, but $f_{yx}(0,0) = +1$.

Operators and Exact Equations

Definition (Operators and Annihilators)

Let $C^1(S) = \{\textit{continuously differentiable fcns on } S\}$.

- An *operator* on S is a fcn $\Phi : C^1(S) \to C^1(S)$.
- An *annihilator* is an operator combination that maps a fcn to 0.

Definition (Exact Differential Equations)

A differential equation $M\,dx + N\,dy = 0$ is *exact* iff there is a function $f(x, y)$ s.t. $M = \partial f/\partial x$ and $N = \partial f/\partial y$.

Examples

- $D_j = \frac{\partial}{\partial x_j}$ is an operator on $C^1(\mathbb{R}^n)$.
- $L = (D-2)^2$ annihilates the function $f_a(x) = axe^{2x}$.
- The DE $(2xy + y^2)dx + (x^2 + 2xy)dy = 0$ is exact from $f(x, y) = x^2y + x\,y^2$.

Partial Antiderivatives and Exact Equations

Example

Solve the DE: $2xy\,dx + (x^2-1)\,dy = 0$

Solution: Set $M = 2xy$ and $N = x^2 - 1$.

1. Since $f_x = M = 2xy$, then $f(x, y) = \int 2xy\,dx = x^2y + \phi(y)$.
   partial antiderivative
2. Now $f_y = N = (x^2 - 1)$, so
   $$\frac{\partial}{\partial y}\left[x^2y + \phi(y)\right] = x^2 - 1.$$
   Since $\frac{\partial}{\partial y}\left[x^2y + \phi(y)\right] = x^2 + \frac{d}{dy}\phi(y)$, we have $\phi'(y) = -1$.
   Whence $\phi(y) = -y$

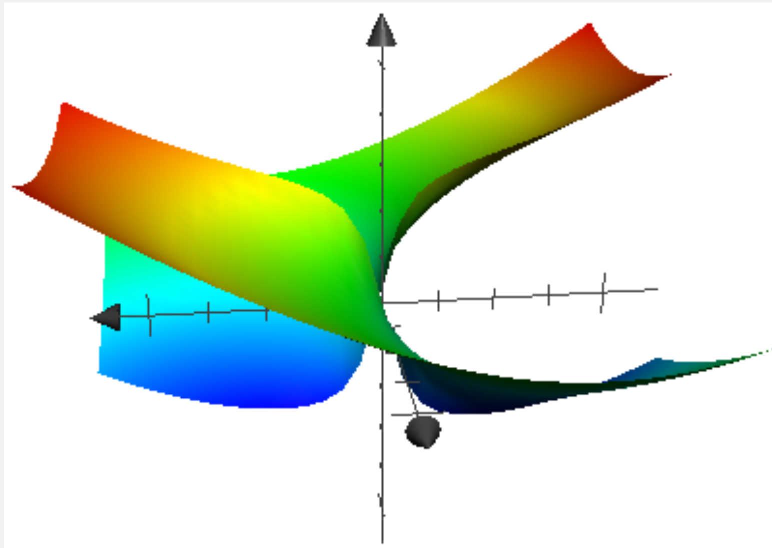
Putting the pieces together, $f(x, y)$ is given by

$$x^2y - y = c$$

where c is a constant of integration.

Try: $\big(x + y/(x^2+y^2)\big)\,dx + \big(y - x/(x^2+y^2)\big)\,dy = 0.$

Picture Time Again



$$f(x,y) = \frac{1}{2}(x^2+y^2) + \arctan\left(\frac{x}{y}\right)$$

Tangent Plane

Consider...

In $\mathbb{R}^2$

- Slope of the tangent line at $x = a$ is $f'(a)$
- Tangent line is $y = f(a) + f'(a)(x-a)$

In $\mathbb{R}^3$

- Tangent vector in the x direction at $\vec{a}$ is $T_x = \langle 1, 0, f_x(\vec{a})\rangle$
- Tangent vector in the y direction at $\vec{a}$ is $T_y = \langle 0, 1, f_y(\vec{a})\rangle$
- A plane containing $\vec{a}$ and the tangent vectors is

$$(T_x \times T_y)\cdot(\vec{x}-\vec{a}) = 0$$

or (with $\vec{a} = \langle x_0, y_0\rangle$ and $\vec{m}_{\vec{a}} = \langle f_x(\vec{a}), f_y(\vec{a})\rangle$)

$$\begin{aligned} z &= f(\vec{a}) + f_x(\vec{a})(x-x_0) + f_y(\vec{a})(y-y_0) \\ &= f(\vec{a}) + \vec{m}_{\vec{a}}\cdot(\vec{x}-\vec{a}) \end{aligned}$$

Differentiation

Definition (Derivative)

Let f be defined on the open set $D \subseteq \mathbb{R}^2$. Then f is *differentiable at* $\vec{x}_0 \in D$ iff there is a vector $\vec{m}$ s.t.

▸ Picture Time

$$f(\vec{x}_0 + \vec{h}) = f(\vec{x}_0) + \vec{m} \cdot \vec{h} + \varepsilon \|\vec{h}\|$$

Equivalently: iff there is a vector $\vec{m}$ s.t. for $T(\vec{x}) = f(\vec{x}_0) + \vec{m} \cdot (\vec{x} - \vec{x}_0)$, then

$$\lim_{\vec{x} \to \vec{x}_0} \frac{f(\vec{x}) - T(\vec{x})}{\|\vec{x} - \vec{x}_0\|} = 0$$

Definition (Gradient)

The *gradient* (*vector*) of f, written as ∇f of $\operatorname{grad}(f)$ is

$$\nabla f(\vec{x}_0) = \left\langle \frac{\partial f}{\partial x} \vec{x}_0, \frac{\partial f}{\partial y} \vec{x}_0 \right\rangle$$

Note: ∇ is a vector differential operator (generalizing D_x): $\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \right\rangle$.

[3] $T(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$

Derivative

Nota Bene

f is differentiable[4] at $\vec{a}$	$\Longrightarrow$	$\frac{\partial f}{\partial x}(\vec{a})$ and $\frac{\partial f}{\partial y}(\vec{a})$ both exist
$\frac{\partial f}{\partial x}(\vec{a})$ and $\frac{\partial f}{\partial y}(\vec{a})$ both exist	$\not\Longrightarrow$	f is differentiable at $\vec{a}$

Theorem (The "Continuity of Partials Suffices" Thm)

If

1. f_x *and* f_y *exist on* $B(\vec{a}; \varepsilon)$ *for some* $\varepsilon > 0$, *and*
2. f_x *and* f_y *are continuous at* $\vec{a}$,

then

1. f *is differentiable at* $\vec{a}$, *and*
2. $f(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a}) + \langle \varepsilon_1, \varepsilon_2 \rangle \cdot (\vec{x} - \vec{a})$
   where $\varepsilon_1, \varepsilon_2 \to 0$ *as* $x - a_x, y - a_y \to 0$, *resp.*

[4] Careful: Gradient is $\nabla = \langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y} \rangle$; Total derivative $f'(\vec{x}_0)$ is $\nabla f(\vec{x}_0)$

Derivative

Proof (The "Continuity of Partials Suffices" Thm).

Let $\vec{a} = \langle x_0, y_0 \rangle$.
NTS: $\Delta f(\vec{a}) = \nabla f(\vec{a}) \cdot \langle \Delta x, \Delta y \rangle + \vec{\varepsilon} \cdot \langle \Delta x, \Delta y \rangle$ with $\vec{\varepsilon} \to \vec{0}$ as $\Delta x, \Delta y \to 0$.

1. Fix y. MVT $\Rightarrow \exists x_1 \in B(x_0; r)$ s.t. $f(x, y) - f(x_0, y) = f_x(x_1, y)(x - x_0)$
2. $f_x \in C(D) \Rightarrow f_x(x_1, y) = f_x(x_0, y_0) + \varepsilon_x$ where $\varepsilon_x \to 0$ as $(x, y) \to (x_0, y_0)$
   So $f(x, y) - f(x_0, y) = [f_x(x_0, y_0) + \varepsilon_x](x - x_0)$ where $\varepsilon_x \underset{x,y \to x_0,y_0}{\longrightarrow} 0$.
3. Fix x. MVT $\Rightarrow \exists y_1 \in B(y_0; r)$ s.t. $f(x, y) - f(x, y_0) = f_y(x, y_1)(y - y_0)$
4. $f_y \in C(D) \Rightarrow f_y(x, y_1) = f_y(x_0, y_0) + \varepsilon_y$ where $\varepsilon_y \to 0$ as $(x, y) \to (x_0, y_0)$
   So $f(x, y) - f(x, y_0) = [f_y(x_0, y_0) + \varepsilon_y](y - y_0)$ where $\varepsilon_y \underset{x,y \to x_0,y_0}{\longrightarrow} 0$.

Whence

$$f(x, y) - f(x_0, y_0) = [f(x, y) - f(x_0, y)] + [f(x_0, y) - f(x_0, y_0)]$$

$$= [f_x(x_0, y_0) + \varepsilon_x](x - x_0) + [f_y(x_0, y_0) + \varepsilon_y](y - y_0)$$

□

Derivatives and Continuity

Theorem ($D \Rightarrow C$ Thm)

If f is differentiable at $\vec{a}$, then f is continuous at $\vec{a}$.

Proof.

Since f is differentiable at $\vec{a}$,

$$f(\vec{a} + \vec{h}) - f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{h} + \vec{\varepsilon} \, \|\vec{h}\|$$

where $\vec{\varepsilon} \to 0$ as $\vec{h} \to 0$. Thus

$$\left| f(\vec{a} + \vec{h}) - f(\vec{a}) \right| \leq \left| \nabla f(\vec{a}) \cdot \vec{h} \right| + |\vec{\varepsilon}| \, \|\vec{h}\|$$

$$\leq \|\nabla f(\vec{a})\| \, \|\vec{h}\| + |\vec{\varepsilon}| \, \|\vec{h}\| = (\|\nabla f(\vec{a})\| + |\vec{\varepsilon}|) \, \|\vec{h}\|$$

Whence $\lim\limits_{\vec{x} \to \vec{a}} f(\vec{x}) = f(\vec{a})$. □

Algebra of Derivatives

Proposition (Algebra of Derivatives)

Let f and g be differentiable functions at $\vec{a}$. Then

- *$f \pm g$ is differentiable at $\vec{a}$*
- *$f \cdot g$ is differentiable at $\vec{a}$*
- *$f \div g$ is differentiable at $\vec{a}$ as long as $g(\vec{a}) \neq 0$*
- $\nabla(f \pm g) = (\nabla f) \pm (\nabla g)$
- $\nabla(f \cdot g) = (\nabla f)g + f(\nabla g)$
- $\nabla(f \div g) = \dfrac{(\nabla f)g - f(\nabla g)}{g^2}$ *when $g(\vec{a}) \neq 0$*

Proof.

Homework. Pg 462, #14. □

See: §10.2. Problem 4, pg461 (Maple time.)

Vector Calculus | Functions of Two Variables | Multiple Integration | Intro to Lebesgue Measure

Directional Derivatives

Thinking Out Loud...

1.
   - f_x is the derivative in the $\langle 1, 0\rangle$ direction
   - f_y is the derivative in the $\langle 0, 1\rangle$ direction
2.
   - $(x_0 + h, y_0) \underset{h\to 0}{\longrightarrow} (x_0, y_0)$ equiv to $\langle x_0, y_0\rangle + h\langle 1, 0\rangle \underset{h\to 0}{\longrightarrow} \langle x_0, y_0\rangle$
   - $(x_0, y_0 + k) \underset{k\to 0}{\longrightarrow} (x_0, y_0)$ equiv to $\langle x_0, y_0\rangle + k\langle 0, 1\rangle \underset{k\to 0}{\longrightarrow} \langle x_0, y_0\rangle$
3. With an arbitrary direction $\vec{u}$ (unit vector): $\vec{x} + h\,\vec{u} \underset{h\to 0}{\longrightarrow} \vec{x}_0$

Definition (Directional Derivative)

Let f be defined on an open set D and $\vec{a} \in D$. Then the *directional derivative of f in the direction of $\vec{u}$, a unit vector*, is given, if the limit is finite, by

$$D_{\vec{u}} f(\vec{a}) = \lim_{h\to 0} \frac{f(\vec{a} + h\,\vec{u}) - f(\vec{a})}{h}$$

or

$$\frac{\partial f}{\partial \vec{u}}(\vec{a}) = \lim_{h\to 0} \frac{f(x + hu_x, y + hu_y) - f(x, y)}{h}$$

Directional Derivative's Properties

Theorem

If f is differentiable at $\vec{a}$, then $D_{\vec{u}}f(\vec{a})$ exists for any direction $\vec{u}$. And

$$D_{\vec{u}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u}$$

Proof.

Simple computation from: $f(\vec{a} + h\vec{u}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (h\vec{u}) + \varepsilon\|h\vec{u}\|$ □

Corollary ("Method of Steepest Ascent/Descent")

Let f be differentiable at $\vec{a}$. Then

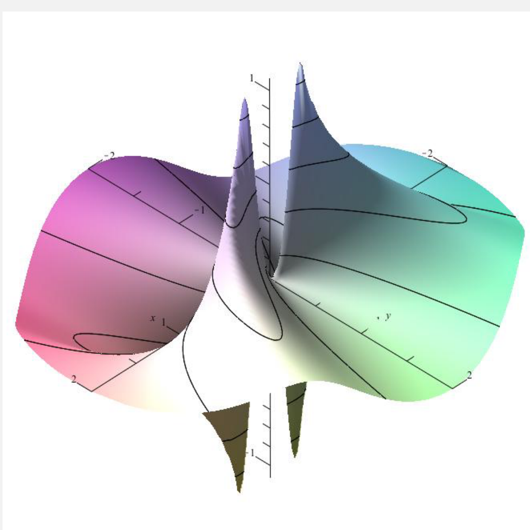
1. *The max rate of change of f at $\vec{a}$ is $\|\nabla f(\vec{a})\|$ in the direction of $\nabla f(\vec{a})$.*
2. *The min rate of change of f at $\vec{a}$ is $-\|\nabla f(\vec{a})\|$ in the direction of $-\nabla f(\vec{a})$.*

Proof.

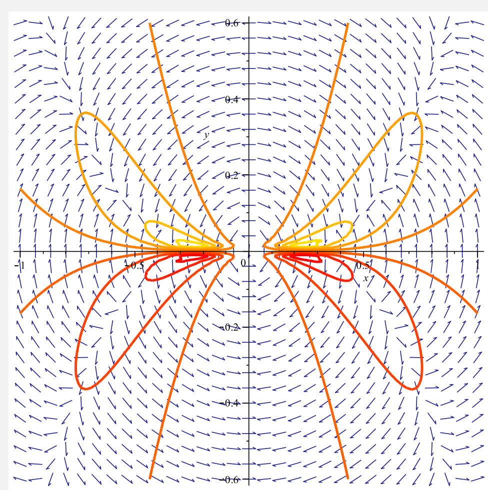
Simple computation from: $D_{\vec{u}}f(\vec{a}) = \nabla f(\vec{a}) \cdot \vec{u} = \|\nabla f(\vec{a})\| \, \|\vec{u}\| \cos(\theta)$ □

Visit Maple.

Directional Derivative's Weird Properties



$$f(x,y) = \frac{x^2 y}{x^6 + y^2}$$



Gradient field & contour plot

f is not continuous at $\vec{0}$, but has directional derivatives in all directions at $\vec{0}$!

The Chain Rule

Theorem (The Chain Rule)

If $x(t)$ and $y(t)$ are differentiable at t_0, and f is differentiable at $\vec{a} = (x(t_0), y(t_0))$, then f composed with x and y is differentiable at t_0 with

$$\frac{d}{dt} f(x(t), y(t)) = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

Proof.

Let $z = f(x, y)$ and $\Delta t = t_1 - t_0$. Then $\Delta x = x(t_1) - x(t_0)$ and $\Delta y = y(t_1) - y(t_0)$. Since f is differentiable, we have

$$\Delta z = f(x + \Delta x, y + \Delta y) - f(x, y) = f_x \Delta x + f_y \Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

So

$$\frac{\Delta z}{\Delta t} = f_x \frac{\Delta x}{\Delta t} + f_y \frac{\Delta y}{\Delta t} + \varepsilon_1 \frac{\Delta x}{\Delta t} + \varepsilon_2 \frac{\Delta y}{\Delta t}$$

Since $\Delta t \to 0 \implies \Delta x, \Delta y \to 0$, then $\varepsilon_1, \varepsilon_2 \to 0$ with Δt. □

The Chain Rule Extended

Corollary (*MCR Corollary*)

If $x(t, s)$ and $y(t, s)$ are differentiable at (t_0, s_0), and $z = f(x, y)$ is differentiable at $\vec{a} = (x(t_0, s_0), y(t_0, s_0))$, then f composed with x and y is differentiable at (t_0, s_0) with

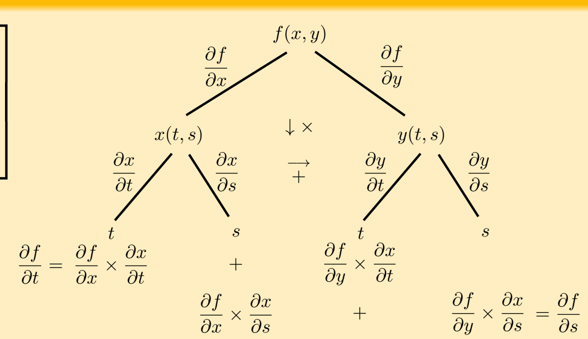
$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial t} \quad \text{and} \quad \frac{dz}{ds} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s}$$

Two Views

$$\begin{bmatrix} \frac{dz}{dt} & \frac{dz}{ds} \end{bmatrix} = \begin{bmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \end{bmatrix} \cdot \begin{bmatrix} \frac{\partial x}{\partial t} & \frac{\partial x}{\partial s} \\ \frac{\partial y}{\partial t} & \frac{\partial y}{\partial s} \end{bmatrix}$$

$$= \nabla f(x, y) \cdot \frac{\partial(x, y)}{\partial(t, s)}$$

$$= \nabla f(x, y) \cdot J_{(x,y)}(t, s)$$



The Mean Value Theorem

Theorem (*MVT for Two*)

Suppose f is differentiable on the open D containing the segment $L(\vec{p},\vec{q})$. Then there is a $\vec{c}$ on L s.t.

$$f(\vec{p}) - f(\vec{q}) = \nabla f(\vec{c}) \cdot (\vec{p} - \vec{q})$$

Proof.

1. Set $(x_0, y_0) = \vec{q}$ and $(h,k) = \vec{p} - \vec{q}$
2. Set $g(t) = f(x_0 + ht, y_0 + kt)$ for $t \in [0,1]$ (g parametrizes f on L)
3. Then $g(1) - g(0) = g'(\theta)(1-0)$ for some $\theta \in (0,1)$; i.e.
$$f(\vec{p}) - f(\vec{q}) = g'(\theta)$$
4. The MCR implies
$$g'(t) = f_x \tfrac{dx}{dt} + f_y \tfrac{dy}{dt} = \langle f_x, f_y \rangle \cdot \langle \tfrac{dx}{dt}, \tfrac{dy}{dt} \rangle$$

□

Taylor's Theorem

Theorem (MV Taylor's Theorem)

Suppose f has partial $(n+1)$st derivatives (of all 'mixtures') existing on $B(\vec{a};r)$. Then for $\vec{x} = \vec{a} + (h,k)$ in $B(\vec{a};r)$,

$$\begin{aligned} f(\vec{a} + (h,k)) =& f(\vec{a}) + \left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right) f(\vec{a}) \\ &+ \frac{1}{2!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^2 f(\vec{a}) + \cdots \\ &+ \frac{1}{n!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^n f(\vec{a}) + R_n \end{aligned}$$

where

$$R_n = \frac{1}{(n+1)!}\left(h\frac{\partial}{\partial x} + k\frac{\partial}{\partial y}\right)^{n+1} f(\vec{a} + \theta(h,k))$$

for some $\theta \in (0,1)$.

Taylor's Theorem Eg

Example (Second Order, Two Variable)

Find the Taylor polynomial of order 2 at $\vec{a} = \langle 1, 1\rangle$ and remainder for $f(x, y) = x^2y$ and $\vec{x} = \langle 1, 1\rangle + \langle h, k\rangle$.

1. $$f(\vec{x}) = f(1,1) + [f_x(1,1)\cdot h + f_y(1,1)\cdot k] + \tfrac{1}{2}\left[f_{xx}(1,1)\cdot h^2 + 2f_{xy}(1,1)\cdot hk + f_{yy}(1,1)\cdot k^2\right] + \tfrac{1}{3!}\left[f_{xxx}(1+\theta h, 1+\theta k)\cdot h^3 + 3f_{xxy}(1+\theta h, 1+\theta k)\cdot h^2k + 3f_{xyy}(1+\theta h, 1+\theta k)\cdot hk^2 + f_{yyy}(1+\theta h, 1+\theta k)\cdot k^3\right]$$
   where $\theta \in (0,1)$
2. $f(1+h, 1+k) = 1 + [2h + k] + \frac{1}{2}\left[2h^2 + 4hk + 0k^2\right] + R_2$

   and $R_2 = \frac{1}{6}\left[0h^3 + 6h^2k + 0hk^2 + 0k^3\right] = h^2k$ with $\theta \in (0,1)$

Multiple Integration

Definition (The Double Sums)

Suppose f is bounded on $R = [a, b] \times [c, d]$. Let $P = P_1 \times P_2$ be a partition of R given by $P_1 = \{a = x_0, \ldots, x_n = b\}$ and $P_2 = \{c = y_0, \ldots, y_m = d\}$ with $R_{ij} = [x_{i-1}, y_{j-1}] \times [x_i, y_j]$. Then the area of R_{ij} is $A_{ij} = \Delta x_i \cdot \Delta y_j$

- Set $\|P\| = \max\{\Delta x_i, \Delta y_j\}$.
- Define
  $$M_{ij}(f) = \sup_{R_{ij}} f(x,y) \quad \text{and} \quad m_{ij}(f) = \inf_{R_{ij}} f(x,y)$$
- Then define
  $$U(P, f) = \sum_i \sum_j M_{ij}\,\Delta x_i \Delta y_j = \sum_{i,j} M_{ij}A_{ij}$$
  $$L(P, f) = \sum_i \sum_j m_{ij}\,\Delta x_i \Delta y_j = \sum_{i,j} m_{ij}A_{ij}$$
  $$S(P, f) = \sum_i \sum_j f(c_i, d_j)\,\Delta x_i \Delta y_j = \sum_{i,j} f(c_i, d_j)A_{ij}$$
  where $(c_i, d_j) \in R_{ij}$ is arbitrary.