A Useful Lemma

Lemma

Let f be bounded on the rectangle R with partition P. Set

$$m = \inf_R f(x,y) \quad \text{and} \quad M = \sup_R f(x,y).$$

1. *Then*
$$m(b-a)(d-c) \le L(P,f) \le S(P,f) \le U(P,f) \le M(b-a)(d-c)$$
2. *If Q partitions R and $P \subseteq Q$, then*
$$L(P,f) \le L(Q,f) \quad \text{and} \quad U(Q,f) \le U(P,f)$$
3. *For any partitions P and Q of R,* $L(P,f) \le U(Q,f)$.
4. $\sup_P L(P,f) \le \inf_P U(P,f)$
5. *The area of R is* $A = \sum_{ij} A_{ij} = (b-a)(d-c)$

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The Integral

Definition (Double Integral)

Let f be bounded on the rectangle R. Then f is *Riemann integrable on* R iff the *upper double integral* and the *lower double integral*, resp.,

$$\overline{\iint}_R f\,dA = \inf_P U(P,f) \quad \text{and} \quad \underline{\iint}_R f\,dA = \sup_P L(P,f)$$

both exist and are equal. We write $\iint_R f\,dA$ for the common value.

Theorem

A bounded function f on the rectangle R is Riemann integrable iff

1. *for any $\varepsilon > 0$ there is a partition P of R s.t.*
$$U(P,f) - L(P,f) < \varepsilon.$$
2. *there is a seq of partitions $\{P_n\}$ s.t.*
$$\lim_{n\to\infty} U(P_n,f) = I = \lim_{n\to\infty} L(P_n,f).$$

A Sample

Example

Find $\displaystyle\iint_R f\,dA$ when $f(x,y) = \frac{1}{2}\sin(x+y)$ and $R = [0, \frac{\pi}{2}]^2$.

1. Use a uniform grid: $x_i = \frac{i}{n}\frac{\pi}{2}$, $y_j = \frac{j}{n}\frac{\pi}{2}$, & $(c_i, d_j) = (x_i, y_j)$ for $i, j = 0..n$
2. A generic Riemann sum becomes

$$S(P_n, f) = \sum_{i,j\in[1,n]} f\left(\tfrac{i}{n}\tfrac{\pi}{2}, \tfrac{j}{n}\tfrac{\pi}{2}\right)\left(\tfrac{i}{n}\tfrac{\pi}{2} - \tfrac{i-1}{n}\tfrac{\pi}{2}\right)\left(\tfrac{j}{n}\tfrac{\pi}{2} - \tfrac{j-1}{n}\tfrac{\pi}{2}\right)$$
$$= \frac{\pi^2}{4n^2}\sum_{i,j\in[1,n]} \tfrac{1}{2}\sin\left(\tfrac{i}{n}\tfrac{\pi}{2} + \tfrac{j}{n}\tfrac{\pi}{2}\right)$$

3. Since $\sin(x+y) = \sin(x)\cos(y) + \cos(x)\sin(y)$, we have

$$S(P_n, f) = \frac{\pi^2}{8n^2}\sum_{i,j\in[1,n]} \left[\sin\left(\tfrac{i}{n}\tfrac{\pi}{2}\right)\cos\left(\tfrac{j}{n}\tfrac{\pi}{2}\right) + \cos\left(\tfrac{i}{n}\tfrac{\pi}{2}\right)\sin\left(\tfrac{j}{n}\tfrac{\pi}{2}\right)\right]$$
$$= \frac{\pi^2}{8n^2}\sum_{i,j\in[1,n]} \left[\sin\left(\tfrac{i}{n}\tfrac{\pi}{2}\right)\cos\left(\tfrac{j}{n}\tfrac{\pi}{2}\right)\right] + \sum_{i,j\in[1,n]}\left[\cos\left(\tfrac{i}{n}\tfrac{\pi}{2}\right)\sin\left(\tfrac{j}{n}\tfrac{\pi}{2}\right)\right]$$

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A Sample (cont)

Example (cont)

4. Distribute the sums

$$S(P_n, f) = \frac{\pi^2}{8n^2}\left[\sum_{i=1}^{n}\sin\left(\tfrac{i}{n}\tfrac{\pi}{2}\right)\sum_{j=1}^{n}\cos\left(\tfrac{j}{n}\tfrac{\pi}{2}\right) + \sum_{i=1}^{n}\cos\left(\tfrac{i}{n}\tfrac{\pi}{2}\right)\sum_{j=1}^{n}\sin\left(\tfrac{j}{n}\tfrac{\pi}{2}\right)\right]$$
$$= 2\frac{\pi^2}{8n^2}\sum_{i=1}^{n}\cos\left(\tfrac{i}{n}\tfrac{\pi}{2}\right)\sum_{j=1}^{n}\sin\left(\tfrac{j}{n}\tfrac{\pi}{2}\right)$$
$$= \left[\frac{\pi}{2n}\sum_{i=1}^{n}\cos\left(\tfrac{i}{n}\tfrac{\pi}{2}\right)\right]\cdot\left[\frac{\pi}{2n}\sum_{j=1}^{n}\sin\left(\tfrac{j}{n}\tfrac{\pi}{2}\right)\right]$$

5. $\displaystyle\lim_{n\to\infty}\frac{\pi}{2n}\sum_{j=1}^{n} T\left(\tfrac{j}{n}\tfrac{\pi}{2}\right) = \int_0^{\pi/2} T(x)\,dx$, so

$$\lim_{n\to\infty} S(P_n, f) = \int_0^{\pi/2}\cos(x)\,dx \cdot \int_0^{\pi/2}\sin(x)\,dx = 1$$

6. Whence $\displaystyle\iint_{[0,\pi/2]\times[0,\pi/2]} \tfrac{1}{2}\sin(x+y)\,dA = 1$

Continuous Functions

Theorem (Continuous Functions Are Integrable)

If f is continuous on $R = [a, b] \times [c, d]$, then f is integrable on R.

Proof.

Let $\varepsilon > 0$. Set $A = \text{area}(R)$.

1. Since f is cont on R, then f is unif cont on R. Hence there is a $\delta > 0$ s.t. whenever $\vec{x_1}, \vec{x_2} \in R$ with $\|\vec{x_1} - \vec{x_2}\| < \delta$, then $|f(\vec{x_1}) - f(\vec{x_2})| < \varepsilon$.
2. Choose a partition P s.t. $\|P\| < \delta$.
3. Then $U(P, f) - L(P, f) = \sum_{i,j} M_{ij} \Delta x_i \Delta y_j - \sum_{i,j} m_{ij} \Delta x_i \Delta y_j$. I.e.,
$$U(P, f) - L(P, f) = \sum_{i,j} (M_{ij} - m_{ij}) \Delta A_{ij} < \sum_{i,j} \varepsilon \Delta A_{ij} = A\,\varepsilon$$

□

Bilinearity

Theorem (Bilinearity of Integration)

1. *Let f_1 and f_2 be integrable on R, and c_1 and c_2 be constants. Then*
$$\iint_R c_1 f_1 \pm c_2 f_2 \, dA = c_1 \iint_R f_1 \, dA \pm c_2 \iint_R f_2 \, dA$$
2. *Let f be bounded on $R = R_1 + R_2$.*
   2.1 *Then f is integrable on R iff f is integrable on R_1 and R_2.*
   2.2 *If f is integrable on R, then*
$$\iint_R f \, dA = \iint_{R_1} f \, dA + \iint_{R_2} f \, dA$$

Proposition

Let f be integrable on R with $m = \min_R f$ and $M = \max_R f$. Then
$$m \cdot \text{area}(R) \le \iint_R f \, dA \le M \cdot \text{area}(R)$$

Iteration

Thinking Out Loud...

1. Fix x^*. Suppose $f(x^*, y)$ is an integrable function of y. Define
$$g(x) = \int_{[c,d]} f(x,y)\,dy$$
Then integrate g to get
$$\int_{[a,b]} \left[\int_{[c,d]} f(x,y)\,dy \right] dx$$
2. Fix y^*. Suppose $f(x, y^*)$ is an integrable function of x. Define
$$h(y) = \int_{[a,b]} f(x,y)\,dx$$
Then integrate h to get
$$\int_{[c,d]} \left[\int_{[a,b]} f(x,y)\,dx \right] dy$$

How do these integrals relate to $\iint_R f\,dA$?

Iteration and Guido Fubini

Theorem (Fubini (1910))

Let f be integrable on a rectangle R. If for each x, the function $h(y) = f(x,y)$ is integrable over $y \in [c,d]$, then $g(x) = \int_c^d f(x,y)\,dy$ is integrable for $x \in [a,b]$, and
$$\iint\limits_R f\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

Corollary

Let f be integrable on a rectangle R. If

1. $h(y) = f(x,y)$ *is integrable over* $y \in [c,d]$, *and*
2. $k(x) = f(x,y)$ *is integrable over* $x \in [a,b]$,

then
$$\iint\limits_R f\,dA = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

Proving Fubini's Theorem

Proof (sketch).

Let $\varepsilon > 0$.

1. Find a partition P of $[a, b] \times [c, d]$ where $U(P, f) - L(P, f) < \varepsilon$
2. 'Slice' this partition into $P_1(x) \times P_2(y)$.
3. Use $U(P_1, g) - L(P_1, g) < U(P, f) - L(P, f)$ to show $g(x) = \int_{[c,d]} f(x, y)dy$ is integrable over $[a, b]$.
4. Show $L(P, f) \leq \int_{[a,b]} g\,dx \leq U(P, f)$
5. Conclude $\int_{[a,b]} g(x)\,dx = \iint_R f(x, y)dA$
6. Use symmetry to have $\int_{[c,d]} h(y)\,dy = \iint_R f(x, y)dA$

Observe the doneness of the proof. □

Fubini Examples

Example (*Good Function! Biscuit!*)

Let $N(x, y) = e^{-(x^2+y^2)}$ and $R = \mathbb{R}^2$.

1. Change to polar coordinates.
$$\iint_R N(x, y)\,dA = \iint_{[0,\infty]\times[0,2\pi]} N(r, \theta)\,dA$$
2. Apply Fubini's thm two ways:
   2.1 $\iint_R N(r, \theta)\,dA = \int_0^{2\pi} \left[\int_0^\infty e^{-r^2} r\,dr\right] d\theta = \int_0^{2\pi} \frac{1}{2}\,d\theta = \pi$
   2.2 $\iint_R e^{-x^2} e^{-y^2} dA = \int_{-\infty}^\infty e^{-y^2}\left[\int_{-\infty}^\infty e^{-x^2} dx\right] dy = \int_{-\infty}^\infty e^{-y^2} dy \cdot \int_{-\infty}^\infty e^{-x^2} dx$
3. Whence $\int_{-\infty}^\infty e^{-x^2} dx = \sqrt{\pi}$. Whereupon $\int_{-\infty}^\infty \frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}x^2} dx = 1$.

Fubini Examples II

Example (*Bad Function! No Biscuit!*)

Let $f(x,y) = \dfrac{x^2 - y^2}{(x^2+y^2)^2}$ on $R = [0,1] \times [0,1]$.

1. $\displaystyle\int_0^1 \left[\int_0^1 f(x,y)\,dx\right] dy = -\frac{\pi}{4}$
2. $\displaystyle\int_0^1 \left[\int_0^1 f(x,y)\,dy\right] dx = +\frac{\pi}{4}$
3. $\displaystyle\int_0^1 \left[\int_0^1 |f(x,y)|\,dy\right] dx = \infty$

So $\displaystyle\iint_R f(x,y)\,dA$ does not exist

The Leibniz Rule

Theorem (Leibniz Rule)

Suppose f *has continuous partials on* $R = [a,b] \times [c,d]$. *Set* $g(x) = \displaystyle\int_c^d f(x,y)\,dy$. *Then* g *is differentiable on* (a,b) *and*

$$\frac{d}{dx}g(x) = \int_c^d \frac{\partial}{\partial x} f(x,y)\,dx$$

Proof.

1. f has cont partials $\implies$ f is cont and differentiable on $\mathrm{int}(R)$
2. Then f is integ., so for every fixed x^*, $f(x^*,y)$ is integ. on $[c,d]$
3. Choose $x \neq x^*$, then $\exists x_0$ between x and x^* s.t.
$$\frac{g(x)-g(x^*)}{x-x^*} = \int_c^d \frac{f(x,y)-f(x^*,y)}{x-x^*}\,dy = \int f_x(x_0,y)\,dy$$
4. Take limits as $x \to x^*$ to finish

□