Camille Jordan's Content

Definition (Jordan Content Zero)

A set S has *Jordan content zero* iff for each $\varepsilon > 0$ there is a finite collection $\mathcal{R}$ of rectangles R_{ij} s.t.

- $S \subseteq \bigcup_{ij} R_{ij}$
- $\operatorname{area}(\mathcal{R}) = \sum_{ij} \operatorname{area}(R_{ij}) < \varepsilon$

A bounded set D is *Jordan measurable* iff ∂D has Jordan content zero.

Examples

- log spiral on $[9.5297^{-1}, 9.5297]$
- unit disk
- Hilbert's plane filling curve, space filling curve

Proposition

- *Rectifiable curves have Jordan content zero.*
- *The union of sets of content zero has content zero.*

Jordan's Extension

Theorem

If f is continuous on $R = [a, b] \times [c, d]$ except on a set of Jordan content zero, then f is integrable on R.

Proof.

1. Since R is compact and f is cont, $\exists M > 0$ s.t. $|f(x, y)| < M$ on R.
2. For each R_{ij} we see $M_{ij} - m_{ij} < 2M$.
3. Let S be the set of discontinuities of f. So S has content zero.
4. Let $\varepsilon > 0$. Find P s.t. for the rect's covering S, the $\sum \operatorname{area}(R_{ij}) < \varepsilon$
5. Divide the P into P_S and $P_{\bar{S}}$ where P_S contains the rectangles covering S. Then $U(P) - L(P) = [U(P_S) + U(P_{\bar{S}})] - [L(P_S) + L(P_{\bar{S}})]$.
6. Combine with 4: $U(P_S) - L(P_S) \leq \sum (M_{ij} - m_{ij})\Delta A_{ij} < 2M\varepsilon$
7. f is unif cont on $P_{\bar{S}}$ so refine P to obtain $M_{ij} - m_{ij} < \varepsilon$ on P'
8. Then $\displaystyle\sum_{R_{ij} \in P'} (M_{ij} - m_{ij})\Delta A_{ij} < \varepsilon \sum \Delta A_{ij} < \varepsilon A$

$\square$

Bounded, Jordan-Measurable Regions

Proposition (Integral on a B'nded, Jordan-Mble Set)

Let D be a bounded, Jordan-measurable region in $\mathbb{R}^2$ and let f be continuous on D. Define $\chi_D(x) = 1$ for $x \in D$ and 0 for $x \notin D$. Suppose the rectangle $R \supset D$.

- $\iint\limits_D f\,dA \triangleq \iint\limits_R f\,\chi_D\,dA$
- *If D is the region $[a,b] \times [\alpha(x), \beta(x)]$ where $\alpha \le \beta$, then*

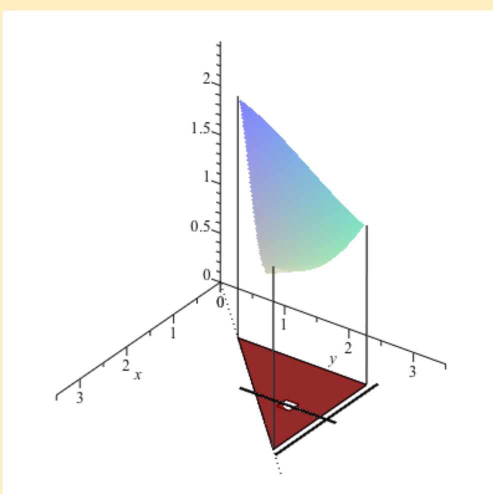
$$\iint\limits_D f\,dA \triangleq \int_a^b \int_{\alpha(x)}^{\beta(x)} f(x,y)\,dy\,dx$$

- *If D is the region $[\alpha(y), \beta(y)] \times [c,d]$ where $\alpha \le \beta$, then*

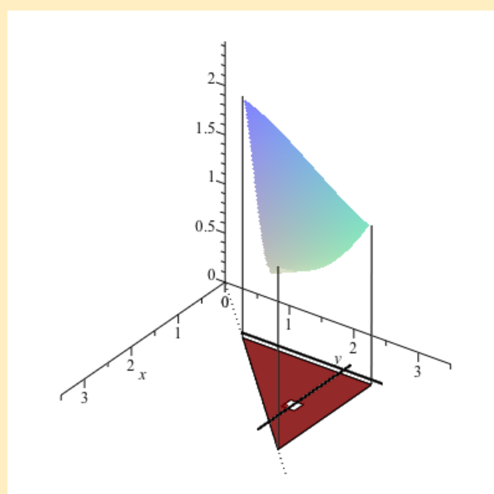
$$\iint\limits_D f\,dA \triangleq \int_c^d \int_{\alpha(y)}^{\beta(y)} f(x,y)\,dx\,dy$$

Dirichlet's Formula

Dirichlet $\subset$ Fubini



$$\int_a^b \int_x^b f(x,y)\,dy\,dx$$



$$\int_a^b \int_a^y f(x,y)\,dx\,dy$$

Line Integrals

Definition (Line Integral)

If f is continuous on a region D containing a smooth curve C, then the *line integral of f along C* is

$$\int_C f\,ds = \lim_{n\to\infty} \sum_{k=1}^{n} f(c_i, d_i)\,\Delta s_i$$

Proposition

If C has a smooth parametrization $(x(t), y(t))$ for $t \in [a, b]$, then

$$\int_C f\,ds = \int_a^b f(x(t), y(t))\,s'(t)\,dt$$

$$= \int_a^b f(x(t), y(t))\,\sqrt{[x'(t)]^2 + [y'(t)]^2}\,dt$$

Line Integrals Are Linear

Proposition (Algebraic Properties)

1. $\displaystyle\int_{-C} f\,ds = -\int_C f\,ds$
2. $\displaystyle\int_C f\,ds = \sum_{i=1}^{n} \int_{C_i} f\,ds$ *where* $C = \bigcup_i C_i$
3. $\displaystyle\left|\int_C f\,ds\right| \le ML$ *where* $L = \text{length}(C)$ *&* $M \ge \max_C |f(x,y)|$.

Examples

1. $\int_C xy\,dx + (x^2 + y^2)dy$ with C the unit circle in the 1st quadrant
2. $\int_C x\,ds$ with C the unit circle in the 1st quadrant
3. $\int_S xy\,dx + (x^2 + y^2)dy$ with S being the unit square having the vertex set $[(1,0), (1,1), (0,1), (0,0)]$

Green's Theorem

Theorem (Green's Theorem[5])

Let D be a simple region in $\mathbb{R}^2$ with a positively-oriented, closed boundary ∂D. If $\vec{F}(x,y) = \langle M(x,y), N(x,y)\rangle$ is a continuously differentiable vector field on an open region containing D, then

$$\oint_{\partial D} M\,dx + N\,dy = \iint_D (N_x - M_y)dx\,dy$$

Theorem (Differential Forms Version)

For D as above and a differentiable $(n-1)$-form ω, $\displaystyle\int_{\partial D} \omega = \int_D d\omega$

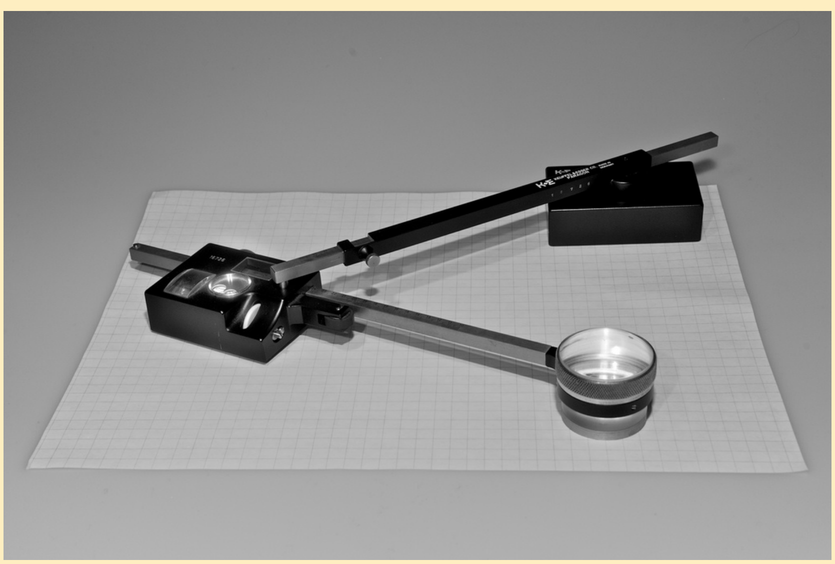
Corollary (Area of a Region)

For f and D as above, $\displaystyle\text{Area}(D) = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx.$

[5] There are a number of equivalent forms of Green's Theorem.

Interlude

Green's Theorem Applied[6]



A Planimeter

Proving Green's Theorem

Proof.

I. $D = \{(x,y) : a \le x \le b$ and $g_1(x) \le y \le g_2(x)\}$. By linearity, NTS:

$$\oint_{\partial D} M\,dx = -\iint_D M_y \quad \text{and} \quad \oint_{\partial D} N\,dy = \iint_D N_x$$

1. Now $\iint_D M_y = \int_a^b \int_{g_1}^{g_2} M_y\,dy\,dx$.
2. The FToC gives $\iint_D M_y = \int_a^b [M(x,g_2) - M(x,g_1)]dx$
3. Decompose ∂D into $D_1 = \{x, g_1(x)\}$, $D_2 = \{x = b, g_1(b) \le y \le g_2(b)\}$, $D_3 = \{x, g_2(x)\}$, and $D_4 = \{x = a, g_2(a) \ge y \ge g_1(a)\}$
4. On D_2 and D_4, $dx = 0$, so $\oint_{\partial D} = \oint_{D_1} + \oint_{D_3}$
5. Then $\oint_{\partial D} M\,dx = \int_a^b M(t, g_1(t))\,dt + \int_b^a M(t, g_2(t))\,dt$
$= \int_a^b M(t, g_1(t)) - M(t, g_2(t))\,dt = -\iint_D M_y$. Aha! $\oint_{\partial D} M\,dx = -\iint_D M_y$.

II. Analogously, $\oint_{\partial D} N\,dy = \iint_D N_x$. □

Forms of Green's Theorem

Theorem

"Under suitable conditions,"

1. $\oint_{\partial D} M\,dx + N\,dy = \oint_{\partial D} \vec{F} \cdot \vec{T}\,ds$ *Circulation Thm*
2. $\oint_{\partial D} M\,dx - N\,dy = \oint_{\partial D} \vec{F} \cdot \vec{N}\,ds$ *Flux Thm*
3. $\iint_D (M_x + N_y)\,dA = \iint_D \mathrm{div}(\vec{F})\,dA$ *Divergence Thm*
4. $\iint_D (N_x - M_y)\,dA = \iint_D \mathrm{curl}(\vec{F})\,dA$ *Curl Thm*

$\mathrm{div}(\vec{v}) = \nabla \cdot \vec{v}$ *and* $\mathrm{curl}(\vec{v}) = \nabla \times \vec{v}$