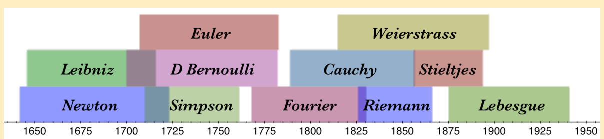
Introduction to Lebesgue Measure

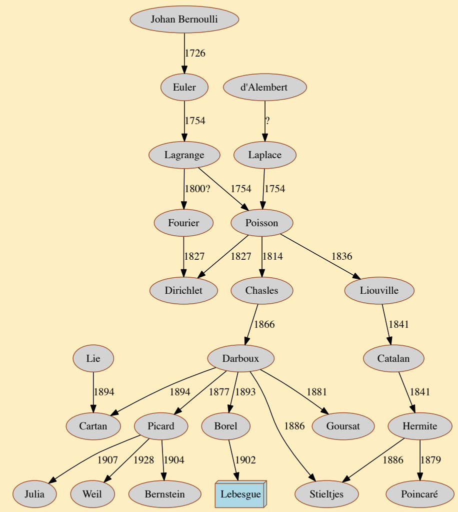
Prelude

There were two problems with calculus: there are functions where

- $f(x) \neq \int f'(x)\,dx$
- $f(x) \neq \dfrac{d}{dx}\left[\int f(x)\,dx\right]$

In his 1902 dissertation, “Intégrale, longueur, aire,” Lebesgue wrote, “It thus seems to be natural to search for a definition of the integral which makes integration the inverse operation of differentiation in as large a range as possible.”





Henri Lebesgue's Mathematical Genealogy (partial)

What’s in a Measure

Goals

THE BEST *measure* would be a real-valued set function μ that satisfies

1. $\mu(I) = \text{length}(I)$ where I is an interval
2. μ is *translation invariant*: $\mu(x+E) = \mu(E)$ for any $x \in \mathbb{R}$
3. if $\{E_n\}$ is pairwise disjoint, then $\mu(\bigcup_n E_n) = \sum_n \mu(E_n)$
4. $\text{dom}(\mu) = \mathcal{P}(\mathbb{R})$ (the power set of $\mathbb{R}$)

THE BAD NEWS:

$$\left\{ \begin{array}{c} \textit{continuum hypothesis} \\ + \;\; \textit{axiom choice} \end{array} \right\} \implies \text{1, 3, and 4 are incompatible}$$

THE PLAN:

- Give up on 4. (cf. *Vitali*)
- 1. and 2. are nonnegotiable
- Weaken 3., then reclaim it

Sigma Algebras

Definition

Sigma Algebra of Sets

Algebra: A collection of sets $\mathcal{A}$ is an *algebra* iff $\mathcal{A}$ is closed under unions and complements.

σ-Algebra: An algebra of sets $\mathcal{A}$ is a *σ-algebra* iff $\mathcal{A}$ is closed under countable unions.

Proposition

Let $\mathcal{A}$ be a nonempty algebra of sets of reals. Then

- $\emptyset$ *and* $\mathbb{R} \in \mathcal{A}$.
- $\mathcal{A}$ *is closed under intersection.*

Let $\mathcal{A}$ be a nonempty σ-algebra of sets of reals. Then

- $\mathcal{A}$ *is closed under countable intersections.*

Sigma Samples

Examples

1. $\mathcal{A} = \{\emptyset, \mathbb{R}\}$
2. $\mathcal{F} = \{F \subset \mathbb{R} : F \text{ is finite or } F^c \text{ is finite}\}$
   2.1 $\mathcal{F}$ is an algebra, the *co-finite algebra*
   2.2 $\mathcal{F}$ is not a σ-algebra
   For each $r \in \mathbb{Q}$, the set $\{r\} \in \mathcal{F}$. But $\bigcup_{r\in\mathbb{Q}}\{r\} = \mathbb{Q} \notin \mathcal{F}$
3. Let $\mathcal{A} = \{\emptyset, [-1,1], (-\infty,-1) \cup (1,\infty), \mathbb{R}\}$. Is $\mathcal{A}$ an algebra?
4. Any intersection of σ-algebras is a σ-algebra
5. Let $\mathcal{B}(\mathbb{R})$ be the smallest σ-algebra containing all the open sets, the *Borel σ-algebra*.

Outer Measure

Definition (Lebesgue Outer Measure)

Let $E \subset \mathbb{R}$. Define the *Lebesgue Outer Measure* μ^* of E to be

$$\mu^*(E) = \inf_{E \subset \bigcup I_n} \sum_n \ell(I_n),$$

the infimum of the sums of the lengths of open interval covers of E.

Proposition (Monotonicity)

If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$.

Proposition

If I is an interval, then $\mu^*(I) = \ell(I)$.

Outer Measure of an Interval

Proof.

I. I is closed and bounded (compact). Then $I = [a, b]$.

1. For any $\varepsilon > 0$, $[a, b] \subset (a - \varepsilon, b + \varepsilon)$. So $\mu^*(I) \leq b - a + 2\varepsilon$. Since ε is arbitrary, $\mu^*(I) \leq b - a$.
2. Let $\{I_n\}$ cover $[a, b]$ with open intervals. There is a finite subcover for $[a, b]$. Order the subcover so that consecutive intervals overlap. Then

$$\sum_N \ell(I_k) = (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_N - a_N)$$

Rearrange

$$\begin{aligned}\sum_N \ell(I_k) &= b_N - (a_N - b_{N-1}) - (a_{N-1} - b_{N-2}) - \cdots - (a_2 - b_1) - a_1 \\ &\geq b_N - a_1 > b - a\end{aligned}$$

Whence $\mu^*(I) = b - a$.

Outer Measure of an Interval, II

Proof (cont).

II. Let I be any bounded interval and $\varepsilon > 0$.

1. There is a closed interval $J \subset I$ so that $\ell(I) - \varepsilon < \ell(J)$. Then

$$\ell(I) - \varepsilon < \ell(J) = \mu^*(J) \leq \mu^*(I) \leq \mu^*(\bar{I}) = \ell(\bar{I}) = \ell(I)$$

III. Suppose I is infinite.

1. Then for each n, there is a closed interval $J \subset I$ s.t. $\ell(J) = n$
2. Thence $\mu^*(I) \geq n$ for all n.

Aha! $\mu^*(I) = \infty$

Proposition

$$\mu^*(\mathbb{Q}) = 0$$

Proof.

Order $\mathbb{Q}$ as $\{r_1, r_2, \ldots\}$. $\{I_n = (r_n - \varepsilon/2^n, r_n + \varepsilon/2^n)\}$ covers $\mathbb{Q}$ $\square$

Countable Subadditivity

Theorem (μ^* is Countably Subadditive)

Let $\{E_n\}$ be a countable set sequence in $\mathbb{R}$. Then $\mu^*\left(\bigcup_n E_n\right) \leq \sum_n \mu^*(E_n)$

Proof.

I. If $\mu^*(E_n) = \infty$ for any n, then done.
II. Let $\varepsilon > 0$

1. For each n find a cover $\{I_{n,j}\}_{n\in\mathbb{N}}$ such that $\sum_{j\in\mathbb{N}} \ell(I_{n,j}) < \mu^*(E_n) + \frac{\varepsilon}{2^n}$
2. Then $\{I_{n,j}\}_{n,j\in\mathbb{N}}$ covers $E = \bigcup_n E_n$.
3. Whereupon

$$\mu^*(E) \leq \sum_{n,j\in\mathbb{N}} \ell(I_{n,j}) = \sum_{n\in\mathbb{N}} \left[\sum_{j\in\mathbb{N}} \ell(I_{n,j})\right]$$

$$< \sum_{n\in\mathbb{N}} \left[\mu^*(E_n) + \frac{\varepsilon}{2^n}\right] = \sum_{n\in\mathbb{N}} [\mu^*(E_n)] + \varepsilon$$

$\square$

Open Holding & Lebesgue's Measure

Corollary

Given $E \subseteq \mathbb{R}$ *and* $\varepsilon > 0$*, there is an open set* $O \supseteq E$ *s.t.*

$$\mu^*(E) \leq \mu^*(O) \leq \mu^*(E) + \varepsilon$$

Definition (Carathéodory's Condition)

A set E is *Lebesgue measurable* iff for every (test) set A,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Let $\mathfrak{M}$ be the collection of all Lebesgue measurable sets.

Corollary

For any A *and* E*,*

$$\mu^*(A) = \mu^*\big((A \cap E) \cup (A \cap E^c)\big) \leq \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Much Ado About Nothing

Theorem

If $\mu^*(E) = 0$*, then* $E \in \mathfrak{M}$*; i.e.,* E *is measurable.*

Proof.

Given the previous corollary, we need only show that

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq \mu^*(A)$$

1. Since $A \cap E \subset E$, then $\mu^*(A \cap E) \leq \mu^*(E) = 0$.
2. Since $A \cap E^c \subset A$, then $\mu^*(A \cap E^c) \leq \mu^*(A)$.

Whence $\mu^*(A \cap E) + \mu^*(A \cap E^c) \leq 0 + \mu^*(A) = \mu^*(A)$. $\square$

Corollary

$$\mu^*(\mathbb{Q}) = 0 \implies \mathbb{Q} \in \mathfrak{M}$$

Unions Work

Theorem

A finite union of measurable sets is measurable.

Proof.

Let E_1 and $E_2 \in \mathfrak{M}$. Let A be a test set.

1. Use $A \cap E_1^c$ as a test set for E_2 which is measurable. Thence
$$\mu^*(A \cap E_1^c) = \mu^*((A \cap E_1^c) \cap E_2) + \mu^*((A \cap E_1^c) \cap E_2^c)$$
2. Note $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^c)$. Whereupon
$$\begin{aligned}
&\mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^c) \\
&\qquad = \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1^c \cap E_2^c)) \\
&\qquad \le \left[\mu^*(A \cap E_1) + \mu^*(A \cap E_2 \cap E_1^c)\right] + \mu^*(A \cap E_1^c \cap E_2^c) \\
&\qquad \le \mu^*(A \cap E_1) + \left[\mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c)\right] \\
&\qquad = \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c) \\
&\qquad = \mu^*(A)
\end{aligned}$$

$\square$

Countable Unions Work

Theorem

The countable union of measurable sets is measurable.

Proof.

Let $E_k \in \mathfrak{M}$ and $E = \bigcup_n E_n$. Choose a test set A.
We need to show $\mu^*(A \cap E) + \mu^*(A \cap E^c) \le \mu^*(A)$.

1. Set $F_n = \bigcup^n E_k$ and $F = \bigcup^\infty E_k = E$. Define $G_1 = E_1$, $G_2 = E_2 - E_1, \ldots, G_k = E_k - \bigcup^{k-1} E_j$, and $G = \bigcup G_k$. Then
(i) $G_i \cap G_j = \emptyset,\ (i \ne j)$ (ii) $F_n = \bigcup^n G_k$ (iii) $F = G = E$
2. Test F_n with A to obtain $\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c)$
3. Test G_n with $A \cap F_n$ to obtain
$$\begin{aligned}
\mu^*(A \cap F_n) &= \mu^*((A \cap F_n) \cap G_n) + \mu^*((A \cap F_n) \cap G_n^c) \\
&= \mu^*(A \cap G_n) + \mu^*(A \cap F_{n-1})
\end{aligned}$$

Countable Unions Work, II

Proof.

4. Iterate $\mu^*(A \cap F_n) = \mu^*(A \cap G_n) + \mu^*(A \cap F_{n-1})$ from 3 to have
$$\mu^*(A \cap F_n) = \sum_{k=1}^{n} \mu^*(A \cap G_k)$$
5. Since $F_n \subseteq F$, then $F^c \subseteq F_n^c$ for all n, then
$$\mu^*(A \cap F_n^c) \geq \mu^*(A \cap F^c)$$
6. Whence
$$\mu^*(A) \geq \sum_{k=1}^{n} \mu^*(A \cap G_k) + \mu^*(A \cap F^c)$$
The summation is increasing & bounded, so convergent.
7. However
$$\sum_{k=1}^{\infty} \mu^*(A \cap G_k) \geq \mu^*\left(\bigcup_{k=1}^{\infty}(A \cap G_k)\right) = \mu^*(A \cap F)$$

Aha! $\mu^*(A) \geq \mu^*(A \cap F) + \mu^*(A \cap F^c)$ $\square$

Everything Works

Corollary

The collection of Lebesgue measurable sets $\mathfrak{M}$ is a σ-algebra.

Corollary

The Borel sets are measurable. (There are measurable, non-Borel sets.)
$$\mathcal{B}(\mathbb{R}) \subsetneqq \mathfrak{M} \subsetneqq \mathcal{P}(\mathbb{R})$$

Definition (Lebesgue Measure)

Lebesgue measure μ is μ^* restricted to $\mathfrak{M}$. So $\mu : \mathfrak{M} \to [0, \infty]$.

Definition (Almost Everywhere)

A property P holds *almost everywhere* (a.e.) iff $\mu(\{x : \neg P(x)\}) = 0$.

The Return of Additivity

Theorem

Let $\{E_n\}$ be a countable (finite or infinite) sequence of pairwise disjoint sets in $\mathfrak{M}$. Then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

Proof.

I. n is finite.

1. For $n = 1$, ✓
2. $\left(\bigcup_{k=1}^{n} E_k\right) \cap E_n = E_n$ and $\left(\bigcup_{k=1}^{n} E_k\right) \cap E_n^c = \bigcup_{k=1}^{n-1} E_k$
3. $\mu\left(\bigcup_{k=1}^{n} E_k\right) = \mu\left(\left[\bigcup_{k=1}^{n} E_k\right] \cap E_n\right) + \mu\left(\left[\bigcup_{k=1}^{n} E_k\right] \cap E_n^c\right)$
   $= \mu(E_n) + \mu\left(\bigcup_{k=1}^{n-1} E_k\right) = \mu(E_n) + \sum_{k=1}^{n-1} \mu(E_k) = \sum_{k=1}^{n} \mu(E_k)$

II. n is infinite.

1. $\bigcup_{k=1}^{n} E_k \subset \bigcup_{k=1}^{\infty} E_k \implies \mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k) \leq \mu\left(\bigcup_{k=1}^{\infty} E_k\right)$
2. A bnded & incr sum converges. Thus $\sum_{k=1}^{\infty} \mu(E_k) \leq \mu\left(\bigcup_{k=1}^{\infty} E_k\right)$
3. Subadditivity finishes the proof.

$\square$

Adding an Example

Example

Set $E_n = \left(\frac{n-1}{n}, \frac{n}{n+1}\right)$ for $n = 1..\infty$.

1. The E_n are pairwise disjoint.
2. $\mu(E_n) = \ell(E_n) = \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)}$
3. $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right]$

Whence $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 1.$

NOTA BENE: $\bigcup_{n=1}^{\infty} E_n = (0,1) - \left\{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\right\}$. Hence $\bigcup_{n=1}^{\infty} E_n = (0,1)$ *a.e.*

Matryoshka

Theorem

If $\{E_n\}$ is a seq of nested, measurable sets with $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

Proof.

1. Set $E = \bigcap_{k=1}^{\infty} E_k$. Set $F_k = E_k - E_{k+1}$. The F_k are pairwise disjoint.
2. Since $\bigcup_{k=1}^{\infty} F_k = E_1 - E$, then $\mu(E_1 - E) = \sum_{k=1}^{\infty} \mu(F_k) = \sum_{k=1}^{\infty} \mu(E_k - E_{k+1})$.
3. If $A \subset B$, then $\mu(A - B) = \mu(A) - \mu(B)$. Apply to the formula above.
4. $\mu(E_1) - \mu(E) = \sum_{k=1}^{\infty} \mu(E_k) - \mu(E_{k+1}) = \mu(E_1) - \lim_{k\to\infty} \mu(E_k)$

Since $\mu(E_1)$ is finite, we're done. □

The Cantor Set

Cantor Sets[7]

I. Constructing C

1. Set $C_0 = [0,1]$
2. Set $C_1 = C_0 - (\frac{1}{3}, \frac{2}{3})$
3. Set $C_2 = C_1 - (\frac{1}{3^2}, \frac{2}{3^2}) - (\frac{7}{3^2}, \frac{8}{3^2})$
4. Set $C_3 = C_2 - (\frac{1}{3^3}, \frac{2}{3^3}) - (\frac{7}{3^3}, \frac{8}{3^3}) - (\frac{19}{3^3}, \frac{20}{3^3}) - (\frac{25}{3^3}, \frac{26}{3^3})$
5. Let $C = \bigcap C_i$

II. Properties of C

1. $\mu(C_0) = 1$, $\mu(C_1) = 2/3$, $\mu(C_2) = 4/9$, $\mu(C_3) = 8/27$, ...So $\mu(C_n) = \frac{2}{3}\mu(C_{n-1}) = \frac{2^n}{3^n}$ Whence $\mu(C) = 0$.
2. C is uncountable
3. C is perfect
4. C is nowhere dense
5. C is compact
6. C is totally disconnected
7. $(\forall i)\ \partial C_i \subset C$
8. $(\forall i)\ \frac{1}{4} \notin \partial C_i$, but $\frac{1}{4} \in C$

Not So Strange After All

Theorem

Let $E \subseteq \mathbb{R}$ and let $\varepsilon > 0$. TFAE:

1. *E is measurable*
2. *There is an open set $O \supset E$ s.t. $\mu^*(O - E) < \varepsilon$*
3. *There is a closed set $F \subset E$ s.t. $\mu^*(E - F) < \varepsilon$*

Proposition

Let S and T be measurable subsets of $\mathbb{R}$. Then

$$\mu(S \cup T) + \mu(S \cap T) = \mu(S) + \mu(T)$$

Functionally Measurable

Theorem (Measurability Conditions for Functions)

Let $f : D \to \mathbb{R}_\infty$ for some $D \in \mathfrak{M}$. TFAE

1. *For each $r \in \mathbb{R}$, the set $f^{-1}\big((r, \infty)\big)$ is measurable.*
2. *For each $r \in \mathbb{R}$, the set $f^{-1}\big([r, \infty)\big)$ is measurable.*
3. *For each $r \in \mathbb{R}$, the set $f^{-1}\big((-\infty, r)\big)$ is measurable.*
4. *For each $r \in \mathbb{R}$, the set $f^{-1}\big((-\infty, r]\big)$ is measurable.*

Proof.

$1 \Rightarrow 2$: $\{x \,|\, f(x) \geq r\} = \bigcap_n \{x \,|\, f(x) > r - 1/n\}$

$2 \Rightarrow 3$: $\{x \,|\, f(x) < r\} = D - \{x \,|\, f(x) \geq r\}$

$3 \Rightarrow 4$: $\{x \,|\, f(x) \leq r\} = \bigcap_n \{x \,|\, f(x) < r + 1/n\}$

$4 \Rightarrow 1$: $\{x \,|\, f(x) > r\} = D - \{x \,|\, f(x) \leq r\}$ $\square$

The Measurably Functional

Corollary

If f satisfies any measurability condition, then $\{x \mid f(x) = r\}$ is measurable for each r.

Definition (Measurable Function)

If a function $f : D \to \mathbb{R}_\infty$ has measurable domain D and satisfies any of the measurability conditions, then f is *measurable*.

Definition

Step function: $\phi : [a, b] \to \mathbb{R}_\infty$ is a *step function* if there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ s.t. ϕ is constant on each interval $I_k = (x_{k-1}, x_k)$, then

$$\phi(x) = \sum_{k=1}^{n} a_k \chi_{I_k}(x)$$

Simple function: A function ψ with range $\{a_1, a_2, \ldots, a_n\}$ where each set $\psi^{-1}(a_k)$ is measurable is a *simple function*.

Simply Stepping

Proposition

Step functions and simple functions are measurable

Theorem (Algebra of Measurable Functions)

Let f and g be measurable on a common domain D, and let $c \in \mathbb{R}$. Then

1. $f + c$
2. $c \cdot f$
3. $f \pm g$
4. f^2
5. $f \cdot g$

are all measurable.

Proof.

- $\checkmark$

$\square$

Sequencing

Theorem

Let $\{f_n\}$ be a sequence of measurable functions on a common domain D. Then

1. $\sup\{f_1, \dots, f_n\}$
2. $\inf\{f_1, \dots, f_n\}$
3. $\sup\limits_{n\to\infty} f_n$
4. $\inf\limits_{n\to\infty} f_n$
5. $\limsup\limits_{n\to\infty} f_n$
6. $\liminf\limits_{n\to\infty} f_n$

are all measurable.

Proof.

1. Set $f = \{f_1, \dots, f_n\}$. Then $\{f(x) > r\} = \bigcup\limits_{k=1}^{n} \{f_k(x) > r\}$.
3. Set $F = \sup_n f_n$. Then $\{F(x) > r\} = \bigcup\limits_{k=1}^{\infty} \{f_k(x) > r\}$.
5. Set $\Phi = \limsup_n f_n$. Then $\limsup\limits_{n\to\infty} f_n = \inf\limits_n \left[\sup\limits_{k\geq n} f_k\right]$

$\square$

Zeroing

Theorem

If f is measurable and $f = g$ a.e., then g is measurable.

Definition (Converence Almost Everywhere)

A sequence $\{f_n\}$ *converges to f almost everywhere*, written as $f_n \to f$ a.e., iff $\mu\Big(\{x : f_n(x) \not\to f(x)\}\Big) = 0$.

Theorem

Let $f : [a, b] \to \mathbb{R}$. Then f is measurable iff there is a seq. of simple functions $\{\psi_n\}$ converging to f a.e.

A Simple Proof

Proof.

($\Rightarrow$) Wolog $f \geq 0$.

1. Define $A_{n,k} = \left\{ x \,\middle|\, \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n} \right\}$ for $k = 1..(n \cdot 2^n)$ and

$$A_{0,n} = [a,b] - \bigcup_{k=1}^{n2^n} A_{n,k}$$

2. Set $\psi_n(x) = n\chi_{A_{0,n}}(x) + \displaystyle\sum_{k=1}^{n2^n} \frac{k-1}{2^n} \cdot \chi_{A_n,k}(x)$
3. Then
   3.1 $\psi_1 \leq \psi_2 \leq \cdots$
   3.2 If $0 \leq f(x) \leq n$, then $|f - \psi_n| < 2^{-n}$
   3.3 $\lim_n \psi = f$ a.e.

($\Leftarrow$) ✓ $\square$

Integration

We began by looking at two examples of integration problems.

- The Riemann integral over $[0,1]$ of a function with infinitely many discontinuities didn't exist even though the points of discontinuity formed a set of measure zero.
  (The points of discontinuity formed a dense set in $[0,1]$.)
- The limit of a sequence of Riemann integrable functions did not equal the integral of the limit function of the sequence.
  (Each function had area $^1/_2$, but the limit of the sequence was the zero function.)

We will look at Riemann integration, then Riemann-Stieltjes integration, and last, develop the Lebesgue integral.

There are many other types of integrals: Darboux, Denjoy, Gauge, Perron, etc. See the list given in the "See also" section of *Integrals* on Mathworld.

Riemann Integral

Definition

- A *partition* $\mathcal{P}$ of $[a,b]$ is a finite set of points such that $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$.
- Set $M_i = \sup f(x)$ on $[x_{i-1}, x_i]$. The *upper sum* of f on $[a,b]$ w.r.t. $\mathcal{P}$ is
$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$$
- The *upper Riemann integral* of f over $[a,b]$ is
$$\overline{\int_a^b} f(x)\,dx = \inf_{\mathcal{P}} U(\mathcal{P}, f)$$

Exercise

1. *Define the lower sum* $L(\mathcal{P}, f)$ *and the lower integral* $\underline{\int_a^b} f$.

Definitely a Riemann Integral

Definition

If $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$, then f is Riemann integrable and is written as $\int_a^b f(x)\,dx$ and $f \in \mathfrak{R}$ on $[a,b]$.

Proposition

A function f is Riemann integrable on $[a,b]$ if and only if for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a,b]$ such that
$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \epsilon.$$

Theorem

If f is continuous on $[a,b]$, then $f \in \mathfrak{R}$ on $[a,b]$.

Theorem

If f is bounded on $[a,b]$ with only finitely many points of discontinuity, then $f \in \mathfrak{R}$ on $[a,b]$.

Properties of Riemann Integrals

Proposition

Let f and $g \in \mathfrak{R}$ on $[a,b]$ and $c \in \mathbb{R}$. Then

- $\int_a^b cf\,dx = c\int_a^b f\,dx$
- $\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$
- $f \cdot g \in \mathfrak{R}$
- *if* $f \le g$, *then* $\int_a^b f\,dx \le \int_a^b g\,dx$
- $\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$
- *Define* $F(x) = \int_a^x f(t)\,dt$. *Then* F *is continuous and, if* f *is continuous at* x_0, *then* $F'(x_0) = f(x_0)$
- *If* $F' = f$ *on* $[a,b]$, *then* $\int_a^b f(x)\,dx = F(b) - F(a)$

Riemann Integrated Exercises

Exercises

1. *If* $\int_a^b |f(x)|\,dx = 0$, *then* $f = 0$.
2. *Show why* $\int_0^1 \chi_{\mathbb{Q}}(x)\,dx$ *does not exist.*
3. *Define*

$$S_n(x) = \sum_{k=1}^{n+1}\left(\frac{k-1}{k}\cdot\chi_{[\frac{k-1}{k},\frac{k}{k+1})}(x)\right) + \frac{n}{n+1}\chi_{[\frac{n+1}{n+2},1]}(x).$$

   3.1 *How many discontinuities does* S_n *have?*
   3.2 *Prove that* $S_n'(x) = 0$ *a.e.*
   3.3 *Calculate* $\int_0^1 S_n(x)\,dx$.
   3.4 *What is* S_∞*?*
   3.5 *Does* $\int_0^1 S_\infty(x)\,dx$ *exist?*

(See an animated graph of S_N.*)*

Riemann-Stieltjes Integral

Definition

- Let $\alpha(x)$ be a monotonically increasing function on $[a, b]$. Set $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.
- Set $M_i = \sup f(x)$ on $[x_{i-1}, x_i]$. The *upper sum* of f on $[a, b]$ w.r.t. α and $\mathcal{P}$ is
$$U(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} M_i \cdot \Delta\alpha_i$$
- The *upper Riemann-Stieltjes integral* of f over $[a, b]$ w.r.t. α is
$$\overline{\int_a^b} f(x)\, d\alpha(x) = \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha)$$

Exercise

1. *Define the lower sum* $L(\mathcal{P}, f, \alpha)$ *and lower integral* $\underline{\int_a^b} f\, d\alpha$.

Definitely a Riemann-Stieltjes Integral

Definition

If $\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$, then f is Riemann-Stieltjes integrable and is written as $\int_a^b f(x)\, d\alpha(x)$ and $f \in \mathfrak{R}(\alpha)$ on $[a, b]$.

Proposition

A function f is Riemann-Stieltjes integrable w.r.t. α on $[a, b]$ iff for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that
$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \epsilon.$$

Theorem

If f is continuous on $[a, b]$, then $f \in \mathfrak{R}(\alpha)$ on $[a, b]$.

Theorem

If f is bounded on $[a, b]$ with only finitely many points of discontinuity and α is continuous at each of f's discontinuities, then $f \in \mathfrak{R}(\alpha)$ on $[a, b]$.

Properties of Riemann-Stieltjes Integrals

Proposition

Let f and $g \in \mathfrak{R}(\alpha)$ and in β on $[a, b]$ and $c \in \mathbb{R}$. Then

- $\int_a^b cf\, d\alpha = c \int_a^b f\, d\alpha$ and $\int_a^b f\, d(c\alpha) = c \int_a^b f\, d\alpha$
- $\int_a^b (f + g)\, d\alpha = \int_a^b f\, d\alpha + \int_a^b g\, d\alpha$ and $\int_a^b f\, d(\alpha + \beta) = \int_a^b f\, d\alpha + \int_a^b f\, d\beta$
- $f \cdot g \in \mathfrak{R}(\alpha)$
- *if* $f \leq g$, *then* $\int_a^b f\, d\alpha \leq \int_a^b g\, d\alpha$
- $\left| \int_a^b f\, d\alpha \right| \leq \int_a^b |f|\, d\alpha$
- *Suppose that* $\alpha' \in \mathfrak{R}$ *and* f *is bounded. Then* $f \in \mathfrak{R}(\alpha)$ *iff* $f\alpha' \in \mathfrak{R}$ *and*
$$\int_a^b f\, d\alpha = \int_a^b f \cdot \alpha'\, dx$$

Riemann-Stieltjes Integrals and Series

Proposition

If f is continuous at $c \in (a, b)$ and $\alpha(x) = r$ for $a \leq x < c$ and $\alpha(x) = s$ for $c < x \leq b$, then

$$\begin{aligned} \int_a^b f\, d\alpha &= f(c)\,(\alpha(c+) - \alpha(c-)) \\ &= f(c)\,(s - r) \end{aligned}$$

Proposition

Let $\alpha = \lfloor x \rfloor$, the greatest integer function. If f is continuous on $[0, b]$, then

$$\int_0^b f(x)\, d\lfloor x \rfloor = \sum_{k=1}^{\lfloor b \rfloor} f(k)$$

Riemann-Stieltjes Integrated Exercises

Exercises

1. $\int_0^1 x\,dx^2$
2. $\int_0^{\pi/2} \cos(x)\,d\sin(x)$
3. $\int_0^{5/2} x\,d(x - \lfloor x \rfloor)$
4. $\int_{-1}^1 e^x\,d|x|$
5. $\int_{-3/2}^{3/2} e^x\,d\lfloor x \rfloor$
6. $\int_{-1}^1 e^x\,d\lfloor x \rfloor$
7. *Set H to be the Heaviside function; i.e.,*

$$H(x) = \begin{cases} 0 & x \leq 0 \\ 1 & \textit{otherwise} \end{cases}.$$

Show that, if f is continuous at 0, then

$$\int_{-\infty}^{+\infty} f(x)\,dH(x) = f(0).$$

Lebesgue Integral

We start with simple functions.

Definition

A function has *finite support* if it vanishes outside a finite interval.

Definition

Let ϕ be a measurable simple function with finite support. If $\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$ is a representation of ϕ, then

$$\int \phi(x)\,dx = \sum_{i=1}^{n} a_i \cdot \mu(A_i)$$

Definition

If E is a measurable set, then $\int_E \phi = \int \phi \cdot \chi_E$.

Integral Linearity

Proposition

If ϕ and ψ are measurable simple functions with finite support and $a, b \in \mathbb{R}$, then $\displaystyle\int (a\phi + b\psi) = a\int \phi + b\int \psi.$ *Further, if $\phi \le \psi$ a.e., then* $\displaystyle\int \phi \le \int \psi.$

Proof (sketch).

I. Let $\phi = \sum^{N} \alpha_i \chi_{A_i}$ and $\psi = \sum^{M} \beta_i \chi_{B_i}$. Then show $a\phi + b\psi$ can be written as $a\phi + b\psi = \sum^{K} (a\alpha_{k_i} + b\beta_{k_j})\chi_{E_k}$ for the properly chosen E_k. Set A_0 and B_0 to be zero sets of ϕ and ψ. (Take $\{E_k : k = 0..K\} = \{A_j \cap B_k : j = 0..N, k = 0..M\}$.)

II. Use the definition to show $\int \psi - \int \phi = \int (\psi - \phi) \ge \int 0 = 0.$ $\square$

Steps to the Lebesgue Integral

Proposition

Let f be bounded on $E \in \mathfrak{M}$ with $\mu(E) < \infty$. Then f is measurable iff

$$\inf_{f \le \psi} \int_E \psi = \sup_{f \ge \phi} \int_E \phi$$

for all simple functions ϕ and ψ.

Proof.

I. Suppose f is bounded by M. Define

$$E_k = \left\{ x : \frac{k-1}{n} M < f(x) \le \frac{k}{n} M \right\}, \qquad -n \le k \le n$$

The E_k are measurable, disjoint, and have union E. Set

$$\psi_n(x) = \frac{M}{n} \sum_{-n}^{n} k\, \chi_{E_k}(x), \qquad \phi_n(x) = \frac{M}{n} \sum_{-n}^{n} (k-1)\, \chi_{E_k}(x)$$

$\square$

SLI (cont)

(proof cont).

Then $\phi_n(x) \leq f(x) \leq \psi(x)$, and so

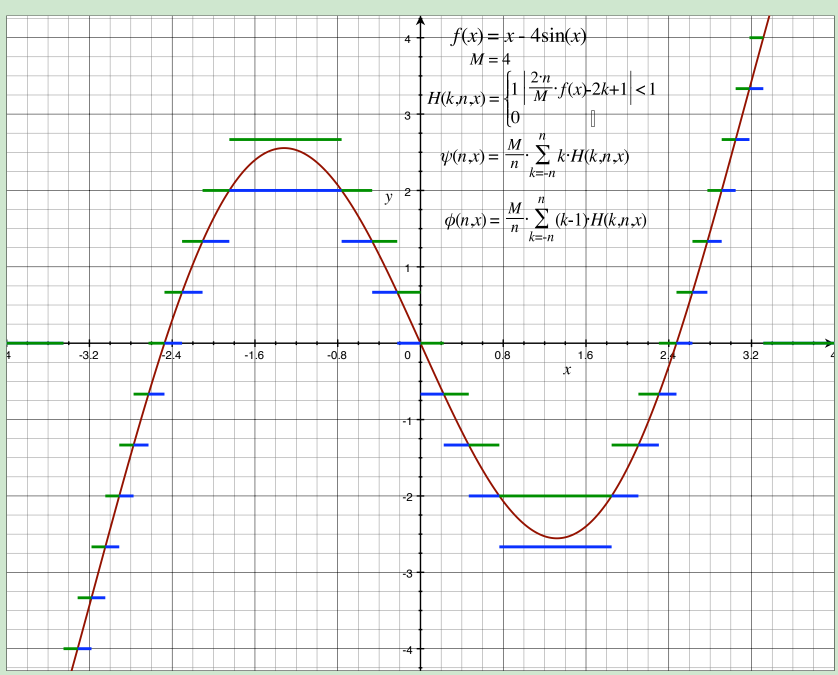
- $\inf \int_E \psi \leq \int_E \psi_n = \frac{M}{n} \sum_{k=-n}^{n} k\, \mu(E_k)$
- $\sup \int_E \phi \geq \int_E \phi_n = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\, \mu(E_k)$

Thus $0 \leq \inf \int_E \psi - \sup \int_E \phi \leq \frac{M}{n}\mu(E)$. Since n is arbitrary, equality holds.

II. Suppose that $\inf \int_E \psi = \sup \int_E \phi$. Choose ϕ_n and ψ_n so that $\phi_n \leq f \leq \psi_n$ and $\int_E(\psi_n - \phi_n) < \frac{1}{n}$. The functions $\psi^* = \inf \psi_n$ and $\phi^* = \sup \phi_n$ are measurable and $\phi^* \leq f \leq \psi^*$. The set $\Delta = \{x : \phi^*(x) < \psi^*(x)\}$ has measure 0. Thus $\phi^* = \psi^*$ almost everywhere, so $\phi^* = f$ a.e. Hence f is measurable. $\square$

Example Steps

Example



Defining the Lebesgue Integral

Definition

If f is a bounded measurable function on a measurable set E with $m(E) < \infty$, then

$$\int_E f = \inf_{\psi \geq f} \int_E \psi$$

for all simple functions $\psi \geq f$.

Proposition

Let f be a bounded function defined on $E = [a, b]$. If f is Riemann integrable on $[a, b]$, then f is measurable on $[a, b]$ and

$$\int_E f = \int_a^b f(x)\, dx;$$

the Riemann integral of f equals the Lebesgue integral of f.

Properties of the Lebesgue Integral

Proposition

If f and g are measurable on E, a set of finite measure, then

- $\displaystyle\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$
- *if $f = g$ a.e., then* $\displaystyle\int_E f = \int_E g$
- *if $f \leq g$ a.e., then* $\displaystyle\int_E f \leq \int_E g$
- $\displaystyle\left| \int_E f \right| \leq \int_E |f|$
- *if $a \leq f \leq b$, then* $\displaystyle a \cdot \mu(E) \leq \int_E f \leq b \cdot \mu(E)$
- *if $A \cap B = \emptyset$, then* $\displaystyle\int_{A \cup B} f = \int_A f + \int_B f$

Lebesgue Integral Examples

Examples

1. Let $T(x) = \begin{Bmatrix} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \\ 0 & \text{otherwise} \end{Bmatrix}$. Then $\int_{[0,1]} T = \int_0^1 T(x)\,dx$.
2. Let $\chi_{\mathbb{Q}}(x) = \begin{Bmatrix} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{Bmatrix}$. Then $\int_{[0,1]} \chi_{\mathbb{Q}} \neq \int_0^1 \chi_{\mathbb{Q}}(x)dx$.
3. Define
$$f_n(x) = \sum_{k=1}^{n+1} \left(\frac{k-1}{k} \cdot \chi_{\left[\frac{k-1}{k}, \frac{k}{k+1}\right)}(x) \right) + \frac{n}{n+1} \chi_{\left[\frac{n+1}{n+2}, 1\right]}(x).$$
Then
   - 3.1 f_n is a step function, hence integrable
   - 3.2 $f_n'(x) = 0$ a.e.
   - 3.3 $\frac{1}{4} \leq \int_{[0,1]} f_n = \int_0^1 f_n(x)\,dx < \frac{3}{8}$

Extending the Integral Definition

Definition

Let f be a nonnegative measurable function defined on a measurable set E. Define
$$\int_E f = \sup_{h \leq f} \int_E h$$
where h is a bounded measurable function with finite support.

Proposition

If f and g are nonnegative measurable functions, then

- $\int_E c\,f = c \int_E f$ *for* $c > 0$
- $\int_E f + g = \int_E f + \int_E g$
- *If $f \leq g$ a.e., then* $\int_E f \leq \int_E g$

General Lebesgue's Integral

Definition

Set $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$. Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$. A measurable function f is integrable over E iff both f^+ and f^- are integrable over E, and then $\int_E f = \int_E f^+ - \int_E f^-$.

Proposition

Let f and g be integrable over E and let $c \in \mathbb{R}$. Then

1. $\int_E cf = c \int_E f$
2. $\int_E f + g = \int_E f + \int_E g$
3. *if $f \leq g$ a.e., then* $\int_E f \leq \int_E g$
4. *if A, B are disjoint m'ble subsets of E,* $\int_{A \cup B} f = \int_A f + \int_B f$

Convergence Theorems

Theorem (Bounded Convergence Theorem)

Let $\{f_n : E \to \mathbb{R}\}$ be a sequence of measurable functions converging to f with $m(E) < \infty$. If there is a uniform bound M for all f_n, then

$$\int_E \lim_n f_n = \lim_n \int_E f_n$$

Proof (sketch).

Let $\epsilon > 0$.

1. f_n converges "almost uniformly;" i.e., $\exists A, N$ s.t. $m(A) < \frac{\epsilon}{4M}$ and, for $n > N, x \in E - A \implies |f_n(x) - f(x)| \leq \frac{\epsilon}{2\,m(E)}$.
2. $\left| \int_E f_n - \int_E f \right| = \left| \int_E f_n - f \right| \leq \int_E |f_n - f| = \left(\int_{E-A} + \int_A \right) |f_n - f|$
3. $\int_{E-A} |f_n - f| + \int_A |f_n| + |f| \leq \frac{\epsilon}{2m(E)} \cdot m(E) + 2M \cdot \frac{\epsilon}{4M} = \epsilon$ $\square$

Lebesgue's Dominated Convergence Theorem

Theorem (Dominated Convergence Theorem)

Let $\{f_n : E \to \mathbb{R}\}$ be a sequence of measurable functions converging a.e. on E with $m(E) < \infty$. If there is an integrable function g on E such that $|f_n| \leq g$ then

$$\int_E \lim_n f_n = \lim_n \int_E f_n$$

Lemma

Under the conditions of the DCT, set $g_n = \sup_{k \geq n} \{f_n, f_{n+1}, \ldots\}$ and $h_n = \inf_{k \geq n} \{f_n, f_{n+1}, \ldots\}$. Then g_n and h_n are integrable and $\lim g_n = f = \lim h_n$ a.e.

Proof of DCT (sketch).

- Both g_n and h_n are monotone and converging. Apply MCT.
- $h_n \leq f_n \leq g_n \implies \int_E h_n \leq \int_E f_n \leq \int_E g_n$. □

Increasing the Convergence

Theorem (Fatou's Lemma)

If $\{f_n\}$ is a sequence of measurable functions converging to f a.e. on E, then

$$\int_E \lim_n f_n \leq \liminf_n \int_E f_n$$

Theorem (Monotone Convergence Theorem)

If $\{f_n\}$ is an increasing sequence of nonnegative measurable functions converging to f, then

$$\int \lim_n f_n = \lim_n \int f_n$$

Corollary (Beppo Levi Theorem (cf.))

If $\{f_n\}$ is a sequence of nonnegative measurable functions, then

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$$

Sidebar: Littlewood's Three Principles

John Edensor Littlewood said,

> *The extent of knowledge required is nothing so great as sometimes supposed. There are three principles, roughly expressible in the following terms:*
>
> - *every measurable set is nearly a finite union of intervals;*
> - *every measurable function is nearly continuous;*
> - *every convergent sequence of measurable functions is nearly uniformly convergent.*
>
> *Most of the results of analysis are fairly intuitive applications of these ideas.*

From *Lectures on the Theory of Functions,* Oxford, 1944, p. 26.

Extensions of Convergence

The sequence f_n converges to f . . .

Definition (Convergence Almost Everywhere)

almost everywhere if $m(\{x : f_n(x) \nrightarrow f(x)\}) = 0$.

Definition (Convergence Almost Uniformly)

almost uniformly on E if, for any $\epsilon > 0$, there is a set $A \subset E$ with $m(A) < \epsilon$ so that f_n converges uniformly on $E - A$.

Definition (Convergence in Measure)

in measure if, for any $\epsilon > 0$, $\lim\limits_{n\to\infty} m\left(\{x : |f_n(x) - f(x)| \geq \epsilon\}\right) = 0$.

Definition (Convergence in Mean (of order $p > 1$))

in mean if $\lim\limits_{n\to\infty} \|f_n - f\|_p = \lim\limits_{n\to\infty} \left[\int_E |f - f_n|^p\right]^{1/p} = 0$

Integrated Exercises

Exercises

1. *Prove: If f is integrable on E, then $|f|$ is integrable on E.*
2. *Prove: If f is integrable over E, then* $\left|\int_E f\right| \leq \int_E |f|$.
3. *True or False: If $|f|$ is integrable over E, then f is integrable over E.*
4. *Let f be integrable over E. For any $\epsilon > 0$, there is a simple (resp. step) function ϕ (resp. ψ) such that* $\int_E |f - \phi| < \epsilon$.
5. *For $n = k + 2^\nu, 0 \leq k < 2^\nu$, define $f_n = \chi_{[k2^{-\nu},(k+1)2^{-\nu}]}$.*
   - 5.1 *Show that f_n does not converge for any $x \in [0, 1]$.*
   - 5.2 *Show that f_n does not converge a.e. on $[0, 1]$.*
   - 5.3 *Show that f_n does not converge almost uniformly on $[0, 1]$.*
   - 5.4 *Show that $f_n \to 0$ in measure.*
   - 5.5 *Show that $f_n \to 0$ in mean (of order 2).*

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