Analysis Comprehensive Sample Exam

Topics include: convergence of uniform convergence for series and sequences of functions, power series, proofs using central notions from calculus (e.g. definitions of limits, continuity, and derivatives, the mean value theorem), Lebesgue measure theory, and vector analysis. Section 1 problems are computational problems on convergence, section 2 problems are more abstract treatments of convergence and proofs from calculus, section 3 problems cover Lebesgue theory, and section 4 covers vector analysis.

Recent exams consisted of several (less than five) problems from each of the four sections. Students were asked to complete two problems from sections 1 through 3 and one problem from section 4. A sample exam (with extra problems) follows.

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Instructions:

> There are four sections in the exam. You should complete two problems in each of the first three parts, and one problem in the fourth.
>
> You may use a calculator and/or Maple.

Sample Problems

Section 1: Complete two problems. All sequences and series range over positive integers. Completely explain your answers.

1. Find the limit of the sequence $\{a_n\}$, where $a_n = n \sin \frac{3}{n}$.
2. Determine whether or not $\sum \frac{2}{k^2+2k}$ converges. If it does, find the sum.
3. Suppose that $f_n(x) = \frac{1}{nx^2+1}$. Determine whether or not the sequence $\{f_n\}$ converges uniformly on $(0, \infty)$.
4. Suppose that $f_n(x) = \frac{1}{n}e^{-n^2x^2}$. Determine whether or not the sequence $\{f_n\}$ converges uniformly on $(0, \infty)$.
5. Let $f_n(x) = \frac{x^n}{x^n+1}$ for $x \in [2, 10]$. If possible, evaluate $\displaystyle\lim_{n\to\infty} \int_3^8 f_n(x)\, dx$.
6. Use an appropriate test to determine if the series $\displaystyle\sum_{n=1}^{\infty} \frac{\sqrt{n+17}}{n^2 - 64n - 112}$ converges or diverges. Justify your answers.
7. For which $p > 0$ does the series $\displaystyle\sum_{n=1}^{\infty} n^p\, p^n$ converge? Justify your answer.
8. For which $p > 0$ does the series $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n\, (\ln n)^p}$ converge? Justify your answer.

9. **Prove** the series $\sum_{n=1}^{\infty}\left(\frac{5}{2^n}-\frac{3}{5^n}\right)$ converges and find the sum.

10. Determine if the series $\sum_{n=1}^{\infty}\frac{(-e)^k}{k^4}$ diverges, converges conditionally, or converges absolutely and justify your answer.

11. Find the pointwise limit of the sequence of functions

$$f_n(x)=\begin{cases}1 & \text{if } x\in[-n,n]\\ 0 & \text{otherwise}\end{cases}$$

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Is the convergence uniform? Prove your answer.

12. Let $f_n(x)=\frac{2nx}{1+n^2x^2}$ where $0\le x\le 1$. Does $f_n(x)$ converge uniformly on $0\le x\le 1$?

13. Prove that if $f_n(x)$ is a sequence of functions such that $\{f_n(x)\}$ converges to $f(x)$ uniformly on $[a,b]$ and each $f_n(x)$ is bounded on $[a,b]$, then $f(x)$ must be bounded on $[a,b]$.

14. Suppose that $f_n(x)=\sum_{k=1}^{n}\frac{1}{k^x}$. Determine whether or not the sequence $\{f_n(x)\}$ converges uniformly on $[a,\infty)$ with $a>1$.

15. Find the radius and interval of convergence of the series $\sum_{k=1}^{\infty}\frac{(2x)^k}{k^2}$.

16. Find the radius and interval of convergence of the series $\sum_{k=1}^{\infty}\frac{(x-4)^k}{k\ 3^k}$.

17. Let $f(x)=\arctan(x)$. Find an infinite power series that converges to $\int_0^{0.1} f(x)\ dx$.

18. Prove $\sum_{k=1}^{\infty}\frac{1}{k^p}$ converges for all $p<1$ and diverges for all $p\ge 1$.

Section 2: Complete two problems.

1. Prove that if $\sum a_k$ converges and $\sum b_k$ diverges, then $\sum (a_k + b_k)$ diverges.
2. Use the $\delta - \varepsilon$ definition of the limit to prove that $\lim_{x \to 0} e^x = 1$.
3. Prove that if f is differentiable at $x = a$, then f is continuous at $x = a$.
4. Use the mean value theorem to prove that $\sin(x) \leq x$ for all $x > 0$.
5. Suppose f is differentiable on (a, b) and f' is bounded. Prove that f is uniformly continuous.
6. Prove the series $\sum_{k=0}^{\infty} r^k$ converges to $\frac{1}{1-r}$ if $|r| < 1$ and diverges if $|r| \geq 1$.
7. Let $\sum_{k=0}^{\infty} a_k$ and $\sum_{k=0}^{\infty} b_k$ be infinite series where $b_k > 0$ for all $k \in \mathbb{N}$. Assume $\sum_{k=0}^{\infty} b_k$ converges and that there exists $n^* \in \mathbb{N}$ such that for all $n > n^*$, we have $|a_n| \leq M\ b_n$ for some positive constant M. Prove $\sum_{k=0}^{\infty} a_k$ converges.
8. Suppose $\sum a_k$ converges absolutely and the sequence $\{b_k\}$ is bounded. Prove $\sum a_k\ b_k$ converges.
9. Suppose $a_k > 0$ and $\sum a_k$ converges. Prove $\sum \frac{1}{a_k}$ diverges.
10. If $\sum_{k=0}^{\infty} a_k$ converges absolutely, then prove $\sum_{k=0}^{\infty} a_k^2$ converges. Is the converse true? Give a proof or counterexample.
11. Suppose $\{f_n(x)\}$ is a sequence of continuous functions that converges uniformly to a function $f(x)$ for all $x \in [a, b]$. Prove $f(x)$ is continuous on $[a, b]$.
12. Suppose $\{f_n(x)\}$ is a sequence of continuous functions that converges uniformly to a function $f(x)$ for all $x \in [a, b]$. Prove
$$\lim_{n \to \infty} \int_a^b f_n(x)\ dx = \int_a^b f(x)\ dx.$$
13. Suppose $\{f_n(x)\}$ is a sequence of continuous functions that converges uniformly to a function $f(x)$ for all $x \in [a, b]$ and that $\{x_n\}$ is a sequence such that $x_n \in [a, b]$ and $\lim_{n \to \infty} x_n = c$. Prove that $\lim_{n \to \infty} f_n(x_n) = f(c)$.
14. Suppose that $\{a_n\}$ is a sequence such that $a_n > 0$ for all n and $\frac{a_{n+1}}{a_n} \leq \alpha$ for all $n \geq n^*$ where $\alpha \in [0, 1)$. Prove the series $\sum a_n$ converges.
15. If $\{a_n\}$ is a sequence that is decreasing and converges to 0, then prove $\sum (-1)^n\ a_n$ converges.

Section 3: Complete two problems.

1. Suppose f is a non-negative measurable function on $[0,1]$. Show that if for every integer $n > 0$, the set $\{x \in [0,1] \mid f(x) > 1/n\}$ has measure zero, then f is equal to zero almost everywhere (that is, $f = 0$ a.e.).

2. Let f be a non-negative measurable function on $[0,1]$. Prove that if the Lebesgue integral of f on $[0,1]$ is 0, then f is equal to zero almost everywhere. (That is, prove that if $\int_{[0,1]} f \, d\mu = 0$ then $f = 0$ a.e.).

3. Suppose that $X \subseteq [0,1]$ and X is a set of measure 0. Prove that $T = \{2x \mid x \in X\}$ is also a set of measure 0.

4. Suppose $S = \{\frac{1}{2^n} \mid n \in \mathbb{N}\}$ and $\tilde{S}$ denotes the complement of S. Define

$$f(x) = \begin{cases} e^x & \text{if } x \in [0,1] \cap \tilde{S} \\ e^{-x} & \text{if } x \in S. \end{cases}$$

   (a) Find the Lebesgue integral of $f(x)$ over $[0,1]$.

   (b) Is $f(x)$ Riemann integrable over $[0,1]$? Justify your answer.

Section 4: Complete one problem.

1. Let S be the surface (with outward normal $\vec{n}$) of the region lying above the rectangle in the $x-y$ plane defined by $0 \leq x \leq 1$ and $0 \leq y \leq 1$, and below the surface $z = x^2+y^2$. Apply the divergence theorem to compute the flux integral $\int\int_S F \cdot \vec{n} \, dS$ given that $F = yz\vec{i} + xz^2\vec{j} + (z+xy)\vec{k}$.

2. Let S be the surface consisting of the portion of the paraboloid $Z = x^2+y^2$ lying below the plane $z = 4$. Suppose $F = -2y\vec{i} + 2x\vec{j} + e^z\vec{k}$. Use Stokes' theorem to calculate the flux $\int\int \nabla \times F \cdot \vec{n} \, dS$ of $\nabla \times F$ over S.

3. Find the area enclosed by the ellipse parameterized by the curve $x = 4\cos(\theta)$, $y = \sin(\theta)$, for $0 \leq \theta \leq 2\pi$, by applying Green's Theorem to the vector field $F = -\frac{1}{2}y\vec{i}+\frac{1}{2}x\vec{j}$.