Lebesgue Integrability and Convergence

Theorem 1 (Bounded Convergence Theorem)**.** *If $\{f_n\}$ is a uniformly bounded sequence of measurable functions converging to f a.e. on $[a,b]$, then f is measurable and*

$$\lim_{n\to\infty}\int_{[a,b]} f_n\,d\mu = \int_{[a,b]}\lim_{n\to\infty} f_n\,d\mu = \int_{[a,b]} f\,d\mu$$

Proof. Wolog $f_n \to f$ on $[a,b]$

1. f_n uniformly bounded (by M) & measurable $\implies$ f bounded & measurable $\implies$ f integrable
2. Let $\varepsilon > 0$. $\exists E$ with $\mu(E) < \varepsilon$ so that $f_n \to f$ uniformly on $([a,b]-E)$ (Egorov (1911))
3. For n large enough

$$\begin{aligned}\left|\int_{[a,b]} f_n\,d\mu - \int_{[a,b]} f\,d\mu\right| &\le \int_{[a,b]} |f_n - f|\,d\mu \\ &= \int_E |f_n - f|\,d\mu + \int_{[a,b]-E} |f_n - f|\,d\mu \\ &\le \int_E 2M\,d\mu + \int_{[a,b]-E} \varepsilon\,d\mu \\ &\le 2m\,\mu(E) + \varepsilon\,(b-a) = K\,\varepsilon.\end{aligned}$$

□

Theorem 2 (Fatou's Lemma (1906))**.** *If $\{f_n\}$ is a sequence of measurable functions converging to f a.e. on E, then*

$$\int_E \lim_n f_n\,d\mu \le \liminf_n \int_E f_n\,d\mu$$

Proof. Wolog $f_n \to f$ on E

1. Choose ψ to be measurable on E, bounded by M, and $\psi \le f$.
2. Define $\psi_n(x) = \min\{f_n(x), \psi(x)\} \le f_n(x)$
3. Then ψ_n is a uniformly bounded sequence of measurable functions, so (BCT)

$$\int_E \lim_{n\to\infty} \psi_n\,d\mu = \lim_{n\to\infty}\int_E \psi_n\,d\mu \le \liminf_{n\to\infty}\int_E f_n\,d\mu$$

4. Take the sup over all bounded, measurable $\psi \le f$:

$$\int_E f\,d\mu \le \liminf_{n\to\infty}\int_E f_n\,d\mu$$

□

Theorem 3 (Beppo Levi's Theorem (1906) / Monotone Convergence Theorem)**.** *If $\{f_n\}$ is a monotone increasing sequence of nonnegative measurable functions converging to f a.e. on $[a,b]$, then f is measurable and*

$$\int_{[a,b]} \lim_n f_n\,d\mu = \lim_n \int_{[a,b]} f_n\,d\mu$$

Proof. Wolog $f_n \to f$ on $[a,b]$

1. Fatou gives

$$\int_{[a,b]} f\,d\mu \le \liminf_{n\to\infty}\int_{[a,b]} f_n\,d\mu$$

2. For each $f_n \le f$, so

$$\limsup_{n\to\infty}\int_{[a,b]} f_n\,d\mu \le \int_{[a,b]} f\,d\mu$$

□

Corollary 4. *If $\{f_n\}$ is a sequence of nonnegative measurable functions, then*

$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$$

Theorem 5 (Lebesgue's Dominated Convergence Theorem (1904))**.** *Let $\{f_n\}$ be a sequence of integrable functions converging to f a.e. on $[a,b]$. If there is an integrable function g on $[a,b]$ such that $|f_n| \leq g$ for all n, then f is integrable on $[a,b]$, and*

$$\lim_{n\to\infty} \int_{[a,b]} f_n \, d\mu = \int_{[a,b]} \lim_{n\to\infty} f_n \, d\mu = \int_{[a,b]} f \, d\mu$$

Proof. Wolog $f_n \to f$ on $[a,b]$

1. f_n integrable $\implies$ f_n measurable $\implies$ f integrable & measurable
2. Define
$$\underline{f}_n = \inf\{f_n, f_{n+1}, \dots\} \quad \text{and} \quad \overline{f}_n = \sup\{f_n, f_{n+1}, \dots\}$$
3. Then $-g \leq \underline{f}_n \leq f_n \leq \overline{f}_n \leq g \implies \underline{f}_n$ and $\overline{f}_n$ are integrable
4. Then $\{g - \overline{f}_n\}$ and $\{g + \underline{f}_n\}$ are nonnegative monotone increasing sequences
5. Therefore $\{g - \overline{f}_n\} \to g - f$ and $\{g + \underline{f}_n\} \to g + f$
6. By Beppo Levi
$$\int_{[a,b]} g + f \, d\mu = \int_{[a,b]} g \, d\mu + \int_{[a,b]} \lim_n \underline{f}_n \, d\mu$$
$$\int_{[a,b]} g - f \, d\mu = \int_{[a,b]} g \, d\mu - \int_{[a,b]} \lim_n \overline{f}_n \, d\mu$$
7. Whence
$$\int_{[a,b]} \lim_n \underline{f}_n \, d\mu = \int_{[a,b]} f \, d\mu = \int_{[a,b]} \lim_n \overline{f}_n \, d\mu$$
8. But $\underline{f}_n \leq f_n \leq \overline{f}_n \implies$
$$\int_{[a,b]} \underline{f}_n \, d\mu \leq \int_{[a,b]} f_n \, d\mu \leq \int_{[a,b]} \overline{f}_n \, d\mu$$

Taking limits obtains the result. □

Criteria for Riemann Integrability

Theorem 6 (Riemann's Criterion for Riemann Integrability)**.** *Let the function f be bounded on the interval $[a,b]$. Then f is Riemann integrable if and only if for any $\varepsilon, \sigma > 0$ there is a $\delta > 0$ such that for any partition P with $\|P\| < \delta$ it follows that*

$$\sum_{\{j \,:\, \omega(f, I_j) > \sigma\}} \Delta x_j < \varepsilon$$

where $I_j = [x_{j-1}, x_j]$.

Theorem 7 (Lebesgue's Criterion for Riemann Integrability)**.** *Let the function f be bounded on the interval $[a,b]$. Then f is Riemann integrable if and only if f is continuous almost everywhere on $[a,b]$.*

Theorem 8 (Lebesgue)**.** *A monotonic function has a derivative almost everywhere.*

Corollary 9. *Monotone functions are Riemann integrable.*

Types of Convergence

Let $\{f_n\}$ be a sequence of functions defined on a common measurable domain $E \subseteq \mathbb{R}$.

Definition 1 (Pointwise). *f_n converges to f* pointwise *on E if and only if ...*

Definition 2 (Uniform). *f_n converges to f* uniformly *on E if and only if ...*

Definition 3 (Almost Everywhere). *f_n converges to f* almost everywhere *("a.e.") on E if and only if $f_n(x) \to f(x)$ except on a set of measure zero.*

Definition 4 (Almost Uniformly). *f_n converges to f* almost uniformly *on E if and only if for every $\varepsilon > 0$ there is a set E_ε of measure less than ε such that $f_n \to f$ uniformly on E_ε^c, the complement of E_ε.*

Definition 5 (In Mean). *f_n converges to f* in mean *on E if and only if*

$$\lim_{n\to\infty} \int_E \left| f_n - f \right| d\mu = 0$$

Definition 6 (In Measure). *f_n converges to f* in measure *on E if and only if for every $\varepsilon > 0$*

$$\lim_{n\to\infty} \mu\left(\{x \in E : \left| f_n(x) - f(x) \right| > \varepsilon\}\right) = 0$$

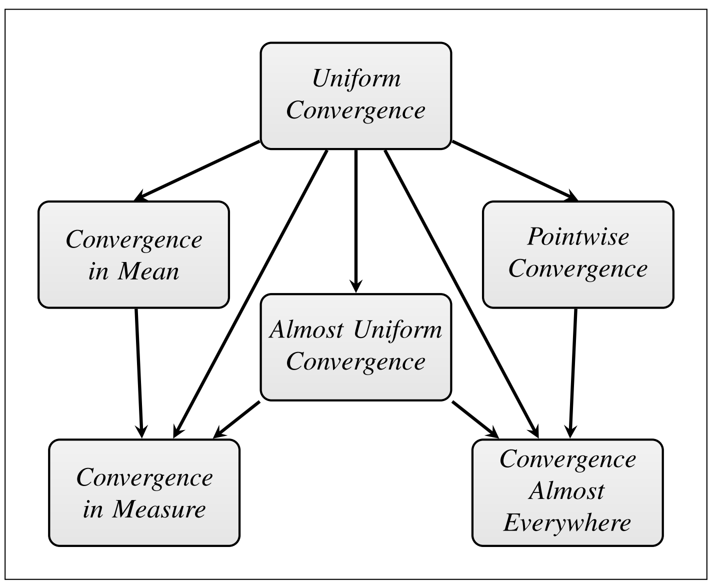

Figure 1: Convergence Hierarchy Chart

Example 1.

Sequence	*Pointwise*	*Uniform*	*Almost Everywhere*	*Almost Uniform*	*In Mean*	*In Measure*
$f_n(x) = \chi_{[n,n+1]}(x)$						
$g_n(x) = \frac{1}{n} \cdot \chi_{[0,n]}(x)$						
$h_n(x) = n \cdot \chi_{[1/n,2/n]}(x)$						
$r_n(x) = \chi_{[j/2^k,(j+1)/2^k]}(x)$ $(n = 2^k + j, j = 0..2^k - 1)$						