Parametric Differentiation and Integration

Example 1. *Integrate* $\int_0^\infty \frac{\sin(x)}{x}\,dx.$

1. Set
$$F(p) = \int_0^\infty e^{-px}\,\frac{\sin(x)}{x}\,dx, \qquad (p \geq 0)$$

2. Differentiate F w.r.t. p
$$F'(p) = -\int_0^\infty e^{-px}\sin(x)\,dx = \int_0^\infty e^{-px}\,d(\cos(x))$$

   Integrate by parts (twice) to get
$$= \frac{-1}{1+p^2}$$

   which yields
$$F(p) = -\arctan(p) + C$$

   Since $\lim_{p\to\infty} F(p) = 0$, then $C = \pi/2$. Thus
$$F(p) = \frac{\pi}{2} - \arctan(p)$$

3. Now
$$F(p) = \frac{\pi}{2} - \arctan(p) = \int_0^\infty e^{-px}\,\frac{\sin(x)}{x}\,dx$$

   implies
$$F(0) = \frac{\pi}{2} - \arctan(0) = \int_0^\infty \frac{\sin(x)}{x}\,dx$$

   I.e.,
$$\int_0^\infty \frac{\sin(x)}{x}\,dx = \frac{\pi}{2}$$

Example 2. *Integrate* $\int_0^\infty e^{-\alpha x^2}\cos(\beta x)\,dx$ *where* $\alpha, \beta > 0$.

1. Set
$$F(\beta) = \int_0^\infty e^{-\alpha x^2}\cos(\beta x)\,dx$$

   Note that
$$F(0) = \int_0^\infty e^{-\alpha x^2}\,dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}$$

2. Now differentiate F w.r.t. β

$$\begin{aligned} F'(\beta) &= -\int_0^\infty x e^{-\alpha x^2} \sin(\beta x)\, dx \\ &= \frac{1}{2\alpha} \int_0^\infty \sin(\beta x)\, d\left(e^{-\alpha x^2}\right) \end{aligned}$$

3. Integrate by parts to have

$$\begin{aligned} F'(\beta) &= \frac{1}{2\alpha} e^{-\alpha x^2} \sin(\beta x)\Big|_0^\infty - \frac{\beta}{2\alpha} \int_0^\infty e^{-\alpha x^2} \cos(\beta x)\, dx \\ &= -\frac{\beta}{2\alpha} \int_0^\infty e^{-\alpha x^2} \cos(\beta x)\, dx = -\frac{\beta}{2\alpha} F(\beta) \end{aligned}$$

4. Solve the initial value problem

$$\begin{cases} F'(\beta) = -\frac{\beta}{2\alpha} F(\beta) \\ F(0) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}} \end{cases}$$

to see

$$F(\beta) = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\, e^{-\frac{\beta^2}{4\alpha}}$$

5. Aha!

$$\int_0^\infty e^{-\alpha x^2} \cos(\beta x)\, dx = \frac{1}{2}\sqrt{\frac{\pi}{\alpha}}\, e^{-\beta^2/(4\alpha)}$$

Example 3. *Integrate the* Laplace integral $\int_0^\infty \frac{\cos(\beta x)}{\alpha^2 + x^2}\, dx$ *for* $\alpha, \beta > 0$.

1. Set

$$L(\beta) = \int_0^\infty \frac{\cos(\beta x)}{\alpha^2 + x^2}\, dx$$

2. Then differentiating w.r.t β gives

$$L'(\beta) = -\int_0^\infty \frac{x \sin(\beta x)}{\alpha^2 + x^2}\, dx$$

3. Differentiating again gives a divergent integral, so instead we'll use partial fractions. Since

$$\frac{\alpha^2 \sin(\beta x)}{x(\alpha^2 + x^2)} = \frac{\sin(\beta x)}{x} - \frac{x \sin(\beta x)}{\alpha^2 + x^2}$$

we have that

$$\begin{aligned} \int_0^\infty \frac{\alpha^2 \sin(\beta x)}{x(\alpha^2 + x^2)}\, dx &= \int_0^\infty \frac{\sin(\beta x)}{x}\, dx - \int_0^\infty \frac{x \sin(\beta x)}{\alpha^2 + x^2}\, dx \\ &= \frac{\pi}{2} + L'(\beta) \end{aligned}$$

Now differentiate w.r.t β again to obtain

$$\alpha^2 L(\beta) = L''(\beta)$$

4. The general solution to this differential equation is $L(\beta) = c_1 e^{\alpha\beta} + c_2 e^{-\alpha\beta}$.
5. Since $L(\beta) \leq \int_0^\infty \frac{dx}{\alpha^2 + x^2} = \frac{\pi}{2\alpha}$, i.e., L is bounded, then $c_1 = 0$. Whence $c_2 = L(0) = \frac{\pi}{2\alpha}$.
6. Aha! $L(\beta) = \int_0^\infty \frac{\cos(\beta x)}{\alpha^2 + x^2}\, dx = \frac{\pi}{2\alpha} e^{-\alpha\beta}$.

Exercises

1. Curiously, $\int_0^\infty e^{-\alpha x^2} \sin(\beta x)\, dx$ does not have an elementary form. Find where the technique fails.
2. Compute $\int_0^1 \frac{x^t - 1}{\ln(x)}\, dx$ for $t > -1$ using D.I. with the parameter t.
3. Look at Talvila's "Some Divergent Trigonometric Integrals," Amer. Math. Monthly **108** (2001), pp 432–436, for an example where Cauchy used this technique improperly.
   (a) What are the integrals Cauchy had wrong?
   (b) What are the appropriate conditions for differentiating under the integral?