MAT 5620	Final Exam	NAME:__________
FALL '14		EMAIL:__________

Work quickly and carefully, following directions closely. Answer all questions completely.

§I. PROBLEMS.

1. Prove or disprove: If $\vec{r}\colon[a,b]\to\mathbb{R}^n$ is integrable, then
$$\left\| \int_a^b \vec{r}(t)\,dt \right\| \leq \int_a^b \|\vec{r}(t)\|\,dt$$

2. Prove: If C is a smooth rectifiable curve, and f is continuous everywhere, then
$$\left| \int_C f\,ds \right| \leq \text{length}(C)\cdot \max_{\vec{x}\in C} |f(\vec{x})|$$

3. Let $\mathscr{S}$ be the surface consisting of the portion of the paraboloid $\Omega = x^2+y^2$ lying below the plane $z=1$. Suppose $F(x,y) = -2y\,\mathbf{i}+2x\,\mathbf{j}+e^z\,\mathbf{k}$. Use Stokes' theorem to calculate the flux $\iint_{\mathscr{S}} \nabla\times F\cdot\vec{n}\,dS$.

4. Let $\psi(x,y) = \begin{cases} \dfrac{y^2-x^2}{(x^2+y^2)^2} & \text{if } (x,y)\neq(0,0) \\ 0 & \text{if } (x,y)=(0,0) \end{cases}$. Show that Fubini's Theorem does not hold for ψ by evaluating
$$\int_0^1\int_0^1 \psi(x,y)\,dx\,dy \quad\text{and}\quad \int_0^1\int_0^1 \psi(x,y)\,dy\,dx.$$
Explain why Fubini's Theorem doesn't hold for ψ on the rectangle $[0,1]\times[0,1]$.

5. Prove: Let E be a subset of N where $\mu(N)=0$. Then $E\in\mathfrak{M}$ and $\mu(E)=0$.

6. Set $f_n(x) = \dfrac{x^n}{1+x^{2n}}$. Apply Lebesgue's *Dominated Convergence Theorem*, if possible, to determine
$$\lim_{n\to\infty}\int_0^\infty f_n\,d\mu$$
showing how the theorem applies or fails to apply.

7. Prove: Let $f:[0,1]\to\mathbb{R}$ be a non-negative measurable function and $\int_{[0,1]} f\,d\mu = 0$. Then $f=0$ almost everywhere.

8. Investigate modes of convergence for the following sequences.
   (a) $g_n:[0,1]\to\mathbb{R}$ given by $g_n(x) = e^n\cdot\chi_{[0,1/n]}(x)$
   (b) $h_n:\mathbb{R}\to\mathbb{R}$ given by $h_n(x) = n^{-1}\cdot\chi_{[0,e^n]}(x)$
   (c) $k_n:[0,\infty)\to\mathbb{R}$ given by $k_n(x) = \chi_{[n,n+1/n]}(x)$

9. Use *parametric integration* to determine the value of the integral $\displaystyle\int_0^1 \frac{x^\tau-1}{\ln(x)}\,dx$ for $\tau>-1$.
(The "elementary calculus approach" substitutes $x=e^{-z}$ for $z=0..\infty$).