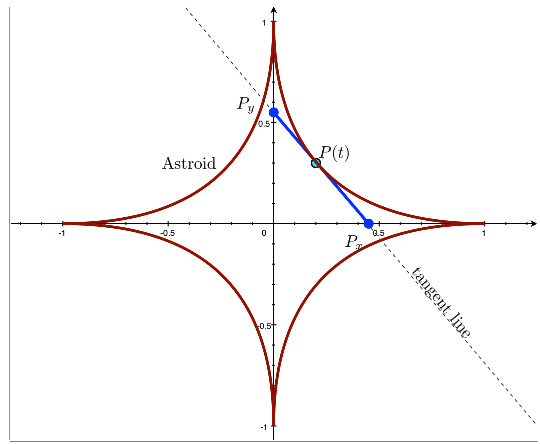
MAT 5620	Midterm Exam	NAME:
FALL '14		EMAIL:

Work quickly and carefully, following directions closely. Answer all questions completely.

§I. PROBLEMS.

1. Let $\mathscr{A}$ be the astroid given by $A(t) = [\cos^3(t), \sin^3(t)]$ for $t \in [0, 2\pi]$. Let $P(t)$ be a point on $\mathscr{A}$. Let P_x and P_y be the x- and y-intercepts of the line tangent to $\mathscr{A}$ at $P(t)$. Show that the line segment $\overline{P_xP_y}$ has constant length; i.e., the length of the segment is independent of t.



2. Let $\vec{r}: \mathbb{R} \to \mathbb{R}^3$ be a vector-valued function that has 2 continuous derivatives for all t. Prove or disprove

$$\frac{d}{dt}[\vec{r}(t) \times \vec{r}\,'(t)] = \vec{r}(t) \times \vec{r}\,''(t).$$

3. Let $f(t) = \dfrac{2t^2}{1+t^2}$ and set $C_{6\pi}$ to be the curve given by $\gamma(t) = [f(t)\cos(2t), f(t)\sin(2t)]$ for $t \in [0, 6\pi]$. Find the length of the curve defined by γ. Make a conjecture concerning the ratio $\dfrac{\text{length}(\gamma_{2n\pi})}{4n}$ as $n \to \infty$.

4. Prove or disprove:
   Let $A_1 = B\left(\begin{bmatrix}1\\0\end{bmatrix}, 1\right)$ and $A_{-1} = B\left(\begin{bmatrix}-1\\0\end{bmatrix}, 1\right)$ be open balls in $\mathbb{R}^2$. Then $E = A_1 \cup A_{-1}$ is not separated.

5. Let $f(x,y) = \begin{cases} \dfrac{xy}{x^2+y^2} & \text{if } (x,y) \neq (0,0) \\ 0 & \text{if } (x,y) = (0,0) \end{cases}$.

   (a) Use polar coordinates to show that $\lim_{(x,y)\to(0,0)} f(x,y)$ does not exist.
   (b) Show that f is not continuous at $(0,0)$.
   (c) If $g(x) = f(x,b)$, where b is any real constant, show that g is continuous at 0.
   (d) Show that $|f(x,y)| \leq \frac{1}{2}$ for all x and y.
   (e) Show $f_x(0,0) = 0 = f_y(,0,0)$ even though f is not continuous at $(0,0)$.

6. A *harmonic function* is one that satisfies *Laplace's equation* $\nabla^2 f(x,y) = 0$ where $\nabla^2 = \dfrac{\partial^2}{\partial x^2} + \dfrac{\partial^2}{\partial y^2}$.

   (a) Prove that the functions
      i. $f(x,y) = x^3 - 3xy^2$
      ii. $g(x,y) = 3x^2y - y^3$

      are harmonic.
   (b) Find $\dfrac{d^2z}{dt^2}$ for $z = f(x,y)$ when $x(t) = \ln(t)$ and $y(t) = e^t$ using the *chain rule*. Don't expand f in terms of t.

7. Find the Taylor expansion centered at the origin to total degree 4 of the *lonely mountain* $\Psi(x,y) = e^{-x^2-y^2}$.