Circular disk

FIGURE 8.1 The disk $B(O; 1)$ is an open set in $\mathbf{R}^2$.

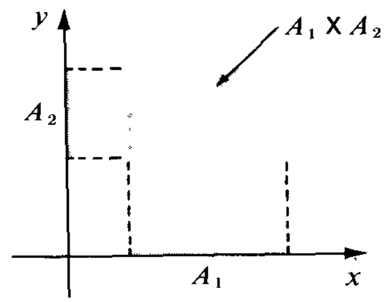

FIGURE 8.2 The Cartesian product of two open intervals is an open rectangle.

of A, x A,. Since A_1 and A_2 are open in $\mathbf{R}^1$ there is a l-ball $B(a_1; r_1)$ in A_1 and a l-ball $B(a_2 ; r_2)$ in A,. Let $r = \min \{r_1, r_2\}$. We can easily show that the 2-ball $B(\boldsymbol{a};$ r) $\subseteq$ A_1 x A,. In fact, if $\boldsymbol{x} = (x_1, x_2)$ is any point of $B(\boldsymbol{a}; r)$ then $\|\boldsymbol{x} - \boldsymbol{a}\| < r$, so $|x_1 - a_1| < r_1$ and $|x_2 - a_2| < r_2$. Hence $x_1 \in B(a_1 ; r_1)$ and $x_2 \in B(a_2 ; r_2)$. Therefore $x_1 \in$ A, and $x_2 \in A_2$, so $(x_1, x_2) \in A_1$ x A,. This proves that every point of $B(\boldsymbol{a}; r)$ is in A, x A_2. Therefore every point of A, x A_2 is an interior point, so A, x A_2 is open.

The reader should realize that an open subset of $\mathbf{R}^1$ is no longer an open set when it is considered as a subset of $\mathbf{R}^2$, because a subset of $\mathbf{R}^1$ cannot contain a 2-ball.

DEFINITIONS OF EXTERIOR AND BOUNDARY. *A point $\boldsymbol{x}$ is said to be exterior to a set S in $\mathbf{R}^n$ if there is an n-ball B(x) containing no points of S. The set of all points in $\mathbf{R}^n$ exterior to S is called the exterior of S and is denoted by* ext S. *A point which is neither exterior to S nor an interior point of S is called a boundary point of S. The set of all boundary points of S is called the boundary of S and is denoted by ∂S.*

These concepts are illustrated in Figure 8.1. The exterior of S is the set of all $\boldsymbol{x}$ with $\|\boldsymbol{x}\| > 1$. The boundary of S consists of all $\boldsymbol{x}$ with $\|\boldsymbol{x}\| = 1$.

8.3 Exercises

1. Letf be a scalar field defined on a set S and let c be a given real number. The set of all points x in S such that $f(\boldsymbol{x}) = c$ is called a *level set* off. (Geometric and physical problems dealing with level sets will be discussed later in this chapter.) For each of the following scalar fields. S is the whole space $\mathbf{R}^n$. Make a sketch to describe the level sets corresponding to the given values of c.
   (a) $f(x, y) = x^2 + y^2$, $\quad c = 0, 1, 4, 9$.
   (b) $f(x, y) = e^{xy}$, $\quad c = e^{-2}, e^{-1}, 1, e, e^2, e^3$.
   (c) $f(x, y) = \cos(x + y)$, $\quad c = -1, 0, \frac{1}{2}, \frac{1}{2}\sqrt{2}, 1$.
   (d) $f(x, y, z) = x + y + z$, $\quad c = -1, 0, 1$.
   (e) $f(x, y, z) = x^2 + 2y^2 + 3z^2$, $\quad c = 0, 6, 12$.
   (f) $f(x, y, z) = \sin(x^2 + y^2 + z^2)$, $\quad c = -1, -\frac{1}{2}, 0, \frac{1}{2}\sqrt{2}, 1$.

2. In each of the following cases, let S be the set of all points (x, y) in the plane satisfying the given inequalities. Make a sketch showing the set S and explain, by a geometric argument, whether or not S is open. Indicate the boundary of S on your sketch.
   (a) $x^2 + y^2 < 1$.
   (b) $3x^2 + 2y^2 < 6$.
   (c) $|x| < 1$ and $|y| < 1$.
   (d) $x \geq 0$ and $y > 0$.
   (e) $|x| \leq 1$ and $|y| \leq 1$.
   (f) $x > 0$ and $y < 0$.
   (g) $xy < 1$.
   (h) $1 \leq x \leq 2$ and $3 < y < 4$.
   (i) $1 < x < 2$ and $y > 0$.
   (j) $x \geq y$.
   (k) $x > y$.
   (l) $y > x^2$ and $|x| < 2$.
   (m) $(x^2 + y^2 - 1)(4 - x^2 - y^2) > 0$.
   (n) $(2x - x^2 - y^2)(x^2 + y^2 - x) > 0$.
3. In each of the following, let S be the set of all points (x, y, z) in 3-space satisfying the given inequalities and determine whether or not S is open.
   (a) $z^2 - x^2 - y^2 - 1 > 0$.
   (b) $|x| < 1$, $|y| < 1$, and $|z| < 1$.
   (c) $x + y + z < 1$.
   (d) $|x| \leq 1$, $|y| < 1$, and $|z| < 1$.
   (e) $x + y + z < 1$ and $x > 0$, $y > 0$, $z > 0$.
   (f) $x^2 + 4y^2 + 4z^2 - 2x + 16y + 40z + 113 < 0$.
4. (a) If A is an open set in n-space and if $x \in A$, show that the set $A - \{x\}$, obtained by removing the point x from A, is open.
   (b) If A is an open interval on the real line and B is a closed subinterval of A, show that $A - B$ is open.†
   (c) If A and B are open intervals on the real line, show that $A \cup B$ and $A \cap B$ are open.
   (d) If A is a closed interval on the real line, show that its complement (relative to the whole real line) is open.
5. Prove the following properties of open sets in $\mathbf{R}^n$:
   (a) The empty set $\varnothing$ is open.
   (b) $\mathbf{R}^n$ is open.
   (c) The union of any collection of open sets is open.
   (d) The intersection of a finite collection of open sets is open.
   (e) Give an example to show that the intersection of an infinite collection of open sets is not necessarily open.

Closed sets. A set S in $\mathbf{R}^n$ is called *closed* if its complement $\mathbf{R}^n - S$ is open. The next three exercises discuss properties of closed sets.

6. In each of the following cases, let S be the set of all points (x, y) in $\mathbf{R}^2$ satisfying the given conditions. Make a sketch showing the set S and give a geometric argument to explain whether S is open, closed, both open and closed, or neither open nor closed.
   (a) $x^2 + y^2 \geq 0$.
   (b) $x^2 + y^2 < 0$.
   (c) $x^2 + y^2 \leq 1$.
   (d) $1 < x^2 + y^2 < 2$.
   (e) $1 \leq x^2 + y^2 \leq 2$.
   (f) $1 < x^2 + y^2 \leq 2$.
   (g) $1 \leq x \leq 2$, $3 \leq y \leq 4$.
   (h) $1 \leq x \leq 2$, $3 \leq y < 4$.
   (i) $y = x^2$.
   (j) $y \geq x^2$.
   (k) $y \geq x^2$ and $|x| < 2$.
   (l) $y \geq x^2$ and $|x| \leq 2$.
7. (a) If A is a closed set in n-space and x is a point not in A, prove that $A \cup \{x\}$ is also closed.
   (b) Prove that a closed interval $[a, b]$ on the real line is a closed set.
   (c) If A and B are closed intervals on the real line, show that $A \cup B$ and $A \cap B$ are closed.

† If A and B are sets, the difference $A - B$ (called the ***complement of B relative to A***) is the set of all elements of A which are not in B.

8. Prove the following properties of closed sets in $\mathbf{R}^n$. You may use the results of Exercise 5.
 (a) The empty set $\varnothing$ is closed.
 (b) $\mathbf{R}^n$ is closed.
 (c) The intersection of any collection of closed sets is closed.
 (d) The union of a finite number of closed sets is closed.
 (e) Give an example to show that the union of an infinite collection of closed sets is not necessarily closed.
9. Let S be a subset of $\mathbf{R}^n$.
 (a) Prove that both int S and ext S are open sets.
 (b) Prove that $\mathbf{R}^n = (\text{int } S) \cup (\text{ext } S) \cup \partial S$, a union of disjoint sets, and use this to deduce that the boundary ∂S is always a closed set.
10. Given a set S in $\mathbf{R}^n$ and a point x with the property that every ball $B(x)$ contains both interior points of S and points exterior to S. Prove that x is a boundary point of S. Is the converse statement true? That is, does every boundary point of S necessarily have this property?
11. Let S be a subset of $\mathbf{R}^n$. Prove that ext $S = \text{int}(\mathbf{R}^n - S)$.
12. Prove that a set S in $\mathbf{R}^n$ is closed if and only if $S = (\text{int } S) \cup \partial S$.

8.4 Limits and continuity

The concepts of limit and continuity are easily extended to scalar and vector fields. We shall formulate the definitions for vector fields; they apply also to scalar fields.

We consider a function f: $S \rightarrow \mathbf{R}^m$, where S is a subset of $\mathbf{R}^n$. If $a \in \mathbf{R}^n$ and $b \in \mathbf{R}^m$ we write

$$\text{(8.1)} \qquad \lim_{x \to a} f(x) = b \qquad (\text{or}, f(x) \to b \text{ as } x \to a)$$

to mean that

$$\text{(8.2)} \qquad \lim_{\|x-a\| \to 0} \| f(x) - b \| = 0 .$$

The limit symbol in equation (8.2) is the usual limit of elementary calculus. In this definition it is not required that f be defined at the point a itself.

If we write $h = x - a$, Equation (8.2) becomes

$$\lim_{\|h\| \to 0} \|f(a + h) - b\| = 0 .$$

For points in $\mathbf{R}^2$ we write (x, y) for x and (a, b) for a and express the limit relation (8.1) as follows :

$$\lim_{(x,y) \to (a,b)} f(x, y) = b.$$

For points in $\mathbf{R}^3$ we put $x = (x, y, z)$ and $a = (a, b, c)$ and write

$$\lim_{(x,y,z) \to (a,b,c)} f(x, y, z) = b .$$

A function f is said to be *continuous* at a if f is defined at a and if

$$\lim_{x \to a} f(x) = f(a) .$$

We say f is *continuous on a set S* iff is continuous at each point of S.