8.5 Exercises

The exercises in this section are concerned with limits and continuity of scalar fields defined on subsets of the plane.

1. In each of the following examples a scalar fieldfis defined by the given equation for all points (x, y) in the plane for which the expression on the right is defined. In each example determine the set of points (x, y) at whichfis continuous.

(a) $f(x, y) = x^4 + y^4 - 4x^2y^2$.

(b) $f(x, y) = \log (x^2 + y^2)$.

(c) $f(x, y) = \frac{1}{y} \cos x^2$.

(d) $f(x, y) = \tan \frac{x^2}{y}$.

(e) $f(x, y) = \arctan \frac{y}{x}$.

(f) $f(x, y) = \arcsin \frac{x}{\sqrt{x^2 + y^2}}$.

(g) $f(x, y) = \arctan \frac{x + y}{1 - xy}$.

(h) $f(x, y) = \frac{x}{\sqrt{x^2 + y^2}}$.

(i) $f(x, y) = x^{(y^2)}$.

(j) $f(x, y) = \arccos \sqrt{x/y}$.

2. If $\lim_{(x,y)\to(a,b)} f(x, y) = L$, and if the one-dimensional limits

$$\lim_{x\to a} f(x, y) \quad \text{and} \quad \lim_{y\to b} f(x, y)$$

both exist, prove that

$$\lim_{x\to a} [\lim_{y\to b} f(x, y)] = \lim_{y\to b} [\lim_{x\to a} f(x, y)] = L.$$

The two limits in this equation are called *iterated* limits; the exercise shows that the existence of the two-dimensional limit and of the two one-dimensional limits implies the existence and equality of the two iterated limits. (Th e converse is not always true. A counter example is given in Exercise 4.)

3. Let $f(x, y) = (x - y)/(x + y)$ if $x + y \neq 0$. Show that

$$\lim_{e-o} [\lim_{y\to 0} f(x, y)] = 1 \quad \text{but that} \quad \lim_{u-o} [\lim_{x\to 0} f(x, y)] = -1.$$

Use this result with Exercise 2 to deduce that $f(x, y)$ does not tend to a limit as $(x, y) \to (0, 0)$.

4. Let

$$f(x, y) = \frac{x^2y^2}{x^2y^2 + (x - y)^2} \quad \text{whenever } x^2y^2 + (x - y)^2 \neq 0.$$

Show that

$$\lim_{x\to 0} [\lim_{y\to 0} f(x, y)] = \lim_{y\to 0} [\lim_{x\to 0} f(x, y)] = 0$$

but that $f(x, y)$ does not tend to a limit as $(x, y) \to (0, 0)$. [*Hint:* Examinefon the line $y = x$.] This example shows that the converse of Exercise 2 is not always true.

5. Let

$$f(x, y) = \begin{cases} x \sin \frac{1}{y} & \text{if } y \neq 0, \\ 0 & \text{if } y = 0. \end{cases}$$

Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ but that

$$\lim_{y\to 0} [\lim_{x\to 0} f(x, y)] \neq \lim_{x\to 0} [\lim_{y\to 0} f(x, y)] .$$

Explain why this does not contradict Exercise 2.

6. If $(x, y) \neq (0, 0)$, let $f(x, y) = (x^2 - y^2)/(x^2 + y^2)$. Find the limit of $f(x, y)$ as $(x, y) \to (0, 0)$ along the line $y = mx$. Is it possible to define $f(0, 0)$ so as to make f continuous at $(0, 0)$?
7. Let $f(x, y) = 0$ if $y \le 0$ or if $y \ge x^2$ and let $f(x, y) = 1$ if $0 < y < x^2$. Show that $f(x, y) \to 0$ as $(x, y) \to (0, 0)$ along any straight line through the origin. Find a curve through the origin along which (except at the origin) $f(x, y)$ has the constant-value 1. Is f continuous at the origin?
8. If $f(x, y) = [\sin (x^2 + y^2)]/(x^2 + y^2)$ when $(x, y) \neq (0, 0)$ how must $f(0, 0)$ be defined so as to make f continuous at the origin?
9. Let f be a scalar field continuous at an interior point a of a set S in $\mathbf{R}^n$. If $f(a) \neq 0$, prove that there is an n-ball $B(a)$ in which f has the same sign as $f(a)$.

8.6 The derivative of a scalar field with respect to a vector

This section introduces derivatives of scalar fields. Derivatives of vector fields are discussed in Section 8.18.

Let f be a scalar field defined on a set S in $\mathbf{R}^n$, and let a be an interior point of S. We wish to study how the field changes as we move from a to a nearby point. For example, suppose $f(a)$ is the temperature at a point a in a heated room with an open window. If we move toward the window the temperature will decrease, but if we move toward the heater it will increase. In general, the manner in which a field changes will depend on the direction in which we move away from a.

Suppose we specify this direction by another vector y. That is, suppose we move from a toward another point $a + y$ along the line segment joining a and $a + y$. Each point on this segment has the form $a + hy$, where h is a real number. An example is shown in Figure 8.3. The distance from a to $a + hy$ is $\|hy\| = |h| \, \|y\|$.

Since a is an interior point of S, there is an n-ball $B(a; r)$ lying entirely in S. If h is chosen so that $|h| \, \|y\| < r$, the segment from a to $a + hy$ will lie in S. (See Figure 8.4.) We keep

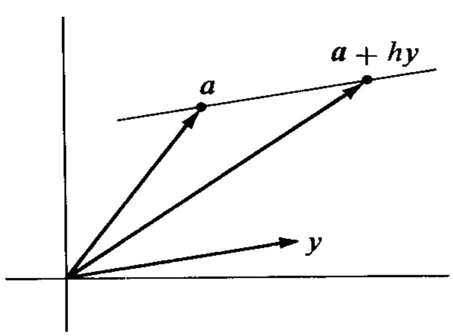

FIGURE 8.3 The point $a + hy$ lies on the line through a parallel to y.

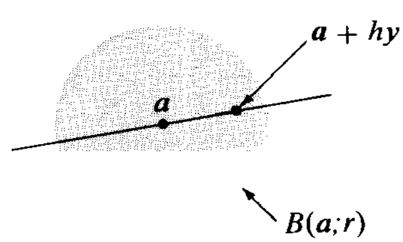

FIGURE 8.4 The point $a + hy$ lies in the n-ball $B(a; r)$ if $\|hy\| < r$.