Properties of the Gradient

Denote the set of functions f from X to Y by Y^X. Then a scalar field $f \in \mathbb{R}^{(\mathbb{R}^n)}$.

Definition 1. *For $f : \mathbb{R}^n \to \mathbb{R}$, define*

$$\nabla f(\vec{x}) = \left\langle \tfrac{\partial}{\partial x_1} f(\vec{x}),\ \tfrac{\partial}{\partial x_2} f(\vec{x}),\ \ldots,\ \tfrac{\partial}{\partial x_n} f(\vec{x}) \right\rangle.$$

Then $\nabla : \mathbb{R}^{(\mathbb{R}^n)} \to (\mathbb{R}^n)^{(\mathbb{R}^n)}$ and as an operator $\nabla = \left\langle \tfrac{\partial}{\partial x_1},\ \tfrac{\partial}{\partial x_2},\ \ldots,\ \tfrac{\partial}{\partial x_n} \right\rangle$.

Proposition 1. *Let $f : \mathbb{R}^n \to \mathbb{R}$.*

1. *If f is differentiable at $\vec{a}$, then all partial derivatives exist and*

$$\begin{aligned} f(\vec{a}; \vec{y}) &= \nabla f(\vec{a}) \cdot \vec{y} \\ &= \|f(\vec{a})\| \cdot \|\vec{y}\| \cdot \cos(\theta) \end{aligned}$$

2. *If all partial derivatives of f exist at $\vec{a}$ and are continuous on an open ball containing $\vec{a}$, then f is differentiable at $\vec{a}$.*

Proposition 2. *Let f be differentiable at $\vec{a}$.*

1. *The first-order Taylor formula for f at $\vec{a}$ is*

$$f(\vec{a} + \vec{v}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot \vec{v} + \|\vec{v}\| \cdot E(\vec{a}, \vec{v})$$

2. *The hyperplane tangent to f at $\vec{a}$ is given by*

$$T(\vec{x}) = f(\vec{a}) + \nabla f(\vec{a}) \cdot (\vec{x} - \vec{a})$$

Theorem 3. *Let f and g be scalar fields that are differentiable on the open set S and let $c \in \mathbb{R}$. Then*

1. *if f is constant, then $\nabla f = \vec{0}$*
2. $\nabla(f + g) = \nabla f + \nabla g$
3. $\nabla(c \cdot f) = c \cdot \nabla f$
4. $\nabla(f \cdot g) = (\nabla f) \cdot g + f \cdot (\nabla g)$
5. $\nabla(f/g) = \dfrac{(\nabla f) \cdot g - f \cdot (\nabla g)}{g^2}$ *where $g \neq 0$*

Theorem 4 (Chain Rule for Scalar Fields)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ and $r : [a, b] \to \mathbb{R}^n$ with $r(t_0) = \vec{a}$. Then*

$$\tfrac{d}{dt}(f \circ r)(t_0) = \nabla f(\vec{a}) \cdot r'(t_0)$$

Proof. (Main Idea:) Consider Δg for t to $t + h$ with f's Taylor form using $\vec{y} = r(t + h) - r(t)$ to obtain

$$\frac{\Delta g}{h} = \nabla f(\vec{a}) \cdot \frac{\Delta r}{h} + \frac{\|\Delta r\|}{h} \cdot E(\vec{a}, \vec{y})$$

Then let $h \to 0$. □