Kepler's Laws

Definition

Planetary motion is planar with the sun in the plane.

Proof.

Set $\vec{r}(t)$ to be the planet's position in space and let $\vec{v}(t) = \vec{r}'(t)$ and $\vec{a}(t) = \vec{v}'(t)$. Then

$$\frac{d}{dt}(\vec{r} \times \vec{v}(t)) = \vec{v} \times \vec{v} + \vec{r} \times \vec{a} = \vec{r} \times \vec{a}$$

Newton's law of gravitation gives $\vec{F} = -GMm/\|\vec{r}\|^2 \cdot (\vec{r}/\|\vec{r}\|)$ and Newton's second law has $\vec{F} = m\vec{a}$, therefore

$$\vec{a} = -\frac{GM}{\|\vec{r}\|^3}\,\vec{r};$$

i.e., $\vec{a}$ is a scalar multiple of $\vec{r}$ so is parallel to $\vec{r}$. Whence $\frac{d}{dt}(\vec{r} \times \vec{v}) = 0$, so $\vec{r} \times \vec{v}$ is a constant. □

Kepler's Laws

Theorem (Kepler's First Law)

In a two-body system (sun and planet), the planet's orbit is an ellipse with the sun at a focus.

Proof.

I. Set $r = \|\vec{r}\|$ and $\vec{u} = \vec{r}/r$. Then $\vec{v} = \frac{d}{dt}(r\vec{u})$ & $\vec{c} = (r\vec{u}) \times \frac{d}{dt}(r\vec{u})$, so $\vec{c} = r^2(\vec{u} \times \frac{d}{dt}\vec{u})$.

II. $\vec{a} \times \vec{c} = [-\frac{GM}{r^2}\vec{u}] \times r^2[\vec{u} \times \frac{d}{dt}\vec{u}] = -GM[\vec{u} \times (\vec{u} \times \frac{d}{dt}\vec{u})] = \cdots = \frac{d}{dt}(GM\vec{u})$.
But $\vec{a} \times \vec{c} = \frac{d}{dt}(\vec{v} \times \vec{c})$, thus $\vec{v} \times \vec{c} = GM\vec{u} + \vec{d}$ for some constant $\vec{d}$.
Now $\vec{u}\cdot\vec{d} = \|\vec{d}\|\cos(\theta)$. Set $c = \|\vec{c}\|$. Then $c^2 = \vec{c}\cdot\vec{c} = r\vec{u}\cdot(GM\vec{u} + \vec{d})$.

III. Hence, $c^2 = GMr + rd\cos(\theta)$, therefore $r = \dfrac{c^2/GM}{1 + (d/(GM))\cos(\theta)}$ which implies $\vec{r}$ is an ellipse when the orbit is closed (parabolic or hyperbolic when not closed). After a good deal of algebraic manipulation, we see that the sun lives at a focus. (Egs. of $r = a/(1 + b\cos(\theta))$: ellipse, parabola, hyperbola.) □

Kepler's Laws

Theorem (Kepler's Second Law)

A planet sweeps through equal areas (wrt the sun) during equal time intervals.

Proof.

We know $A(\theta) = \int_{\theta_0}^{\theta} \frac{1}{2}\, r^2\, d\phi$. Kepler's law becomes $\frac{d}{dt} A = \text{ constant}$. Since

$$\frac{dA}{dt} = \frac{dA}{d\theta}\frac{d\theta}{dt} = \frac{1}{2} r^2(\theta)\, \frac{d\theta}{dt}$$

Recall $\vec{r} = r\vec{u}$. Then $\vec{u} = \cos(\theta)\, \mathbf{i} + \sin(\theta)\, \mathbf{j}$ so that $\frac{du}{dt} = -\sin(\theta)\frac{d\theta}{dt}\, \mathbf{i} + \cos(\theta)\frac{d\theta}{dt}\, \mathbf{j}$. Since $\vec{r} \times \vec{v}$ is constant, $\vec{c} = r^2(\vec{u} \times \frac{d}{dt}\vec{u}) = r^2 \frac{d\theta}{dt}\, \mathbf{k}$. Whence $\frac{dA}{dt} = \frac{1}{2}\|c\|$, a constant. Therefore A is linear wrt t; i.e., $\Delta A = k\Delta t$. □

Kepler's Laws

Theorem (Kepler's Third Law)

Let T be the time to complete one orbit, and a be the length of the orbit's semimajor axis. Then $T^2 \propto a^3$.

Proof.

The area of the orbital ellipse is

$$\pi ab = \int_0^T A'(t)\,dt = \int_0^T \tfrac{1}{2}c\,dt = \tfrac{1}{2}cT \implies T^2 = 4\pi^2 \cdot \frac{a^2b^2}{c^2}$$

where b is the semiminor axis length and c is from the proof of the Second Law. Using the equation of the orbital ellipse (relating a, b, and $c = \|\vec{r} \times \vec{v}\|$) gives

$$T^2 = \left[\frac{4\pi^2}{GM}\right] \cdot a^3$$

□

Kepler's Laws

Timeline

- Aristarchus (c. 300 BC): Earth orbits the Sun.
- Ptolemy (c. 150 AD): Earth is the stationary center of the universe.
- Copernicus gave (1543): orbits are circles, sun is the center, velocity is constant.
- Brahe (1588 & 1598) published his extensive astronomical data.
- Kepler published the laws (c. 1610-1619, mainly based on Brahe's records of the motion of Mars).
- Galileo (1632) publishes *Dialogue Concerning the Two Chief World Systems* and sentenced by the Inquisition for heresy (1633)
- Newton (1687) showed his laws of motion and gravitation implied Kepler's laws.