Limit Superior and Limit Inferior

Definition (Limit Superior)

The *limit superior* of a sequence is the largest subsequential limit; it is written as $\limsup\limits_{n\to\infty} a_n$ or $\varlimsup\limits_{n\to\infty} a_n$.

Proposition

Let $\{a_n\}$ be a sequence and set $\mathscr{A} = \{\text{subsequential limits of } a_n\}$. Then

$$\limsup_{n\to\infty} a_n = \sup \mathscr{A} = \lim_{n\to\infty} [\sup_{k\geq n} a_k].$$

Proposition

Let $\{a_n\}$ be a bounded sequence in $\mathbb{R}$, and let $x \in \mathbb{R}$. Then

1. *if $x > \limsup\limits_{n\to\infty} a_n$, then $\exists k \in \mathbb{N}$ such that if $n \geq k$, then $a_n < x$;*
2. *if $x < \limsup\limits_{n\to\infty} a_n$, then $\forall k \in \mathbb{N}$ there is an $\tilde{n} \geq k$ with $a_{\tilde{n}} > x$.*

Limit Superior, Limit Inferior, and Limit

Proposition

Let $\{a_n\}$ be a sequence and set

$$U = \limsup_{n\to\infty} a_n \quad \text{and} \quad L = \liminf_{n\to\infty} a_n.$$

Then there exist two (at least) subsequences $\{a_{n_k}\}$ and $\{a_{m_j}\}$ such that $a_{n_k} \to U$ and $a_{m_j} \to L$, respectively.

Proposition

Let $\{a_n\}$ be a sequence and set

$$U = \limsup_{n\to\infty} a_n \quad \text{and} \quad L = \liminf_{n\to\infty} a_n.$$

Then either

$U = L$ *in which case $\{a_n\}$ must converge (to $A = U = L$), or*

$U \neq L$ *and then $\{a_n\}$ diverges.*

Properties of Limit Superior and Limit Inferior

Proposition

Let $\{a_n\}$ be a sequence in $\mathbb{R}$. Then

1. $\limsup_{n\to\infty} a_n = \inf_{n\geq 1} [\sup_{k\geq n} x_k]$
2. $\lim_{n\to\infty} a_n = a \iff \liminf_{n\to\infty} a_n = \limsup_{n\to\infty} a_n$
3. $\liminf_{n\to\infty} a_n \leq \limsup_{n\to\infty} a_n$
4. $\limsup_{n\to\infty} (a_n + b_n) \leq \limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$
5. *let* $\limsup a_n = S$. *Then for any* $\varepsilon > 0$,
   - $a_n < S + \varepsilon$ *for all but a finite number of* n*; and*
   - $a_n > S - \varepsilon$ *for infinitely many values of* n.

Problems

Exercises

Set

$$\{a_n\}: \ 0,1,2,1,0,1,2,1,0,1,2,1,0,1,2,1,\ldots$$
$$\{b_n\}: \ 2,1,1,0,2,1,1,0,2,1,1,0,2,1,1,0,\ldots$$

In Exercises 1. to 5., calculate:

1. $\liminf_{n\to\infty} a_n + \liminf_{n\to\infty} b_n$.
2. $\liminf_{n\to\infty} (a_n + b_n)$.
3. $\liminf_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
4. $\limsup_{n\to\infty} (a_n + b_n)$.
5. $\limsup_{n\to\infty} a_n + \limsup_{n\to\infty} b_n$.
6. *Compare the values you found in 1. to 5.*