Green's Theorem and Friends

Theorem 1 (Green's Theorem for Simply Connected Domains (1828))**.** *Let C be a positively oriented, piecewise smooth, simple, closed curve and let D be the region enclosed by the curve. If P and Q have continuous partial derivatives on D, then*

$$\oint_C P\,dx + Q\,dy = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy.$$

Corollary 2 (Planimeter Corollary)**.** *Let C be a positively oriented, piecewise smooth, simple, closed curve and let D be the region enclosed by the curve. Then*

$$\text{Area}(D) = \tfrac{1}{2}\oint_C x\,dy - y\,dx.$$

Theorem 3 (Circulation Theorem)**.** *Let C be a positively oriented, piecewise smooth, simple, closed curve and let D be the region enclosed by the curve. If the vector field $F = \langle P, Q\rangle$ has continuous partial derivatives on D, then*

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_D \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy = \iint_D \text{curl}(\vec{F})\cdot \vec{n}\,dx\,dy.$$

Theorem 4 (Stokes' Theorem (1854))**.** *Let S be a simple parametric surface, $S = r(T)$, where $T \subset \mathbb{R}^2$ is bounded by a piecewise smooth Jordan curve Γ and r is $1 \to 1$ with continuous second-order partials. Set $C = r(\Gamma)$ and $F = \langle P, Q, R\rangle$ be continuously differentiable on S.*

$$\oint_C P\,dx + Q\,dy + R\,dz = \iint_S \left[\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right] dy\wedge dz + \left[\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right] dz\wedge dx + \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\wedge dy$$

where $dx\wedge dy = \dfrac{\partial(x,y)}{\partial(u,v)}\,du\,dv$*, etc; or*

$$\oint_C \vec{F}\cdot d\vec{r} = \iint_T \nabla\times\vec{F}\cdot\vec{n}\,ds.$$

Theorem 5 (Divergence Theorem [Gauss's Theorem (1813)])**.** *Let V be a solid in $\mathbb{R}^3$ bounded by an orientable surface S with unit normal $\vec{n}$. If $\vec{F}$ is continuously differentiable on V, then*

$$\iiint_V \text{div}(\vec{F})\,dx\,dy\,dz = \iint_R \vec{F}\cdot\vec{n}\,dS$$

or

$$\oint_C \vec{F}\cdot\vec{n}\,ds = \iint_D \left(\nabla\cdot\vec{F}\right) dx\,dy.$$

Theorem 6 (Green's Theorem for Multiply Connected Domains)**.** *Let the Jordan curves C_o (outer boundary) and $C_{i_1}, \ldots, C_{i_n}$ (inner boundaries) define a multiply connected region R on which $\vec{F} = \langle P, Q\rangle$ is continuously differentiable. Then*

$$\iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy = \oint_{C_o} P\,dx + Q\,dy - \sum_{k=1}^{n}\oint_{C_{i_k}} P\,dx + Q\,dy.$$

Ref: "The History of Stokes' Theorem," V. Katz, *Mathematics Magazine*, Vol 52, No 3, May 1979, pp 146-156.