Flux Integral

1. Find the flux of the field $\mathbf{F}(x,y) = 2x\mathbf{i} - 3y\mathbf{j}$ over the ellipse $C(t) = [\cos(t), 4\sin(t)]$ for $t \in [0, 2\pi]$.

$$\oint_C \mathbf{F}\cdot\mathbf{n}\,ds = \oint_C M\,dy - N\,dx = \oint_C 2x\,dy + 3y\,dx = -4\pi$$

(Net flux is inward.)

A Green Integral

2. Let $\Gamma(t) = [\cos(2\pi t), \sin(2\pi t)]$ on $t \in [0,1]$. Set $\mathbf{F}(x,y) = -y\,\mathbf{i} + x\,\mathbf{j}$

$$\oint_\Gamma -y\,dx + x\,dy = \int_0^1 2\pi\big(\sin^2(2\pi t) + \cos^2(2\pi t)\big)\,dt = 2\pi$$

and

$$\iint_{x^2+y^2\le 1} 2\,dx\,dy = 2\int_{t=0}^1 \int_{r=0}^1 r\,dr\,dt = 2\pi$$

(Circumference of the unit circle is 2π and the area is π.)

Divergence Integral

3. Find the divergence integral of $\mathbf{F}(x,y,z) = x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}$ over the unit sphere $S : x^2 + y^2 + z^2 = 1$.

$$\operatorname{div}(\mathbf{F}) = \nabla\cdot\mathbf{F} = 3$$

so

$$\iiint_S \operatorname{div}(\mathbf{F})\,dV = 3\iiint_S dV = 4\pi$$

A Stoke's Integral

4. Let H be the positive hemisphere $z = \sqrt{1-x^2-y^2}$ for $0 \le x^2+y^2 \le 1$. Set $\mathbf{F}(x,y) = y\,\mathbf{i} - x\,\mathbf{j}$. Then ∂H is the circle $\{x^2+y^2 = 1, z = 0\}$.

$$\begin{aligned}\mathbf{F}\cdot d\mathbf{r} &= \mathbf{F}\cdot(\mathbf{i}\,dx + \mathbf{j}\,dy + \mathbf{k}\,dz)\\ &= y\,dx - x\,dy\end{aligned}$$

So

$$\oint_{\partial H} \mathbf{F}\cdot d\mathbf{r} = \oint_{\partial H} y\,dx - x\,dy = -8\pi$$

Now $\operatorname{curl}(\mathbf{F}) = \big[P_y - N_z\big]\,\mathbf{i} + \big[M_z - P_x\big]\,\mathbf{j} + \big[N_x - M_y\big]\,\mathbf{k} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ M & N & P \end{vmatrix}$ where $M = y$, $N = -x$, and $P = 0$. Hence $\operatorname{curl}(\mathbf{F}) = -2\mathbf{k}$. The unit normal to H is $\mathbf{n} = \frac{1}{2}\big(x\,\mathbf{i} + y\,\mathbf{j} + z\,\mathbf{k}\big)$. Thus $\operatorname{curl}(\mathbf{F})\cdot\mathbf{n}\,ds = -z\,ds$. Whereupon

$$\iint_H \operatorname{curl}(\mathbf{F})\cdot\mathbf{n}\,ds = \iint_{x^2+y^2\le 1} -2\,dx\,dy = -8\pi$$