Green's Theorem

This theorem is a form of the Fundamental Theorem of Calculus applied to vector fields in the plane. George Green (1793–1841) discovered the theorem in 1828 and published it privately; Mikhail Ostrogradski (1801–1861) discovered it independently three years later. The theorem was virtually unknown until Lord Kelvin [William Thomson (1824–1907)] found Green's paper in 1846 and spread it widely. Green's Theorem led to the Curl Theorem and Stokes' Theorem (actually discovered by Lord Kelvin around 1850) which is needed in the development of Maxwell's Equations. Stokes' Theorem in turn led to the generalized Fundamental Theorem of Calculus. The 4-dimensional Fundamental Theorem is needed for space-time computations and Einstein's General Theory of Relativity.

Theorem 1 (Green's Theorem) *Let D be a bounded region in $\mathcal{R}^2$ with boundary ∂D oriented positively and $\vec{F}(x,y) = [P(x,y), Q(x,y)]$ be a vector field on D. Then*

$$\oint_{\partial D} P\,dx + Q\,dy = \iint_D (Q_x - P_y)\,dx\,dy$$

which, in vector form, is

$$\oint_{\partial D} \vec{F}\cdot d\vec{r} = \iint_D \left(\nabla\times\vec{F}\right)\cdot d\vec{a}$$

Corollary 2 *Let D be a region in $\mathcal{R}^2$. Then*

$$\mathrm{Area}(D) = \frac{1}{2}\oint_{\partial D} x\,dy - y\,dx$$

Problems

1. Apply Green's Thm. to the region inside the square with vertices $(\pm1,\pm1)$ with $\vec{F} = [x^2+y, 2x+2y]$.
2. Apply Green's Theorem to D, the region inside the square with vertices $(\pm1,\pm1)$ and outside the unit circle, with $\vec{F} = [x^2+y, 2x+2y]$.
3. Describe what happens in Green's Theorem if the vector field $\vec{F}$ is *conservative* over D?
4. Calculate the area enclosed by the unit circle using the parametrization $\partial D = [\cos(t), \sin(t)]$ for $t = 0..2\pi$.
5. Discuss the relation of Green's Theorem to the *Flux* and *Circulation integrals*.

Green's Theorem is a special case (2-dimensional) of the Curl Theorem which itself is a special case of Stokes' Theorem. (In the language of *differential forms*, these theorems are all $\int_{\partial D}\omega = \int_D d(\omega)$.)

Theorem 3 (Curl Theorem) *Let S be a surface in $\mathcal{R}^3$ acted on by a vector field $\vec{F}$. Then*

$$\int_{\partial S} \vec{F}\cdot d\vec{r} = \iint_S \mathrm{curl}(\vec{F})\cdot d\vec{S}$$

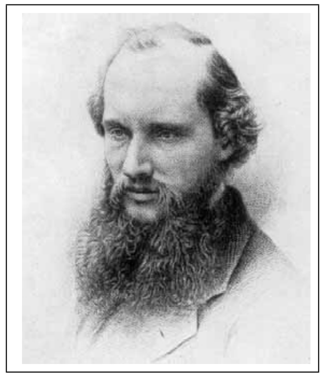
where $\mathrm{curl}(\vec{F}) = \nabla\times\vec{F}$.

Recall that for $\vec{F} = [P(x.y.z), Q(x.y.z), R(x.y.z)]$, we have

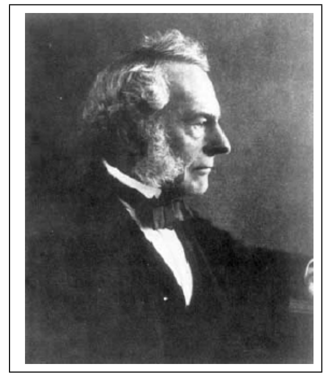
$$\nabla\times\vec{F} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix}$$

People

Mikhail Ostrogradski
(1801–1861)



William Thomson, Lord Kelvin
(1824–1907)

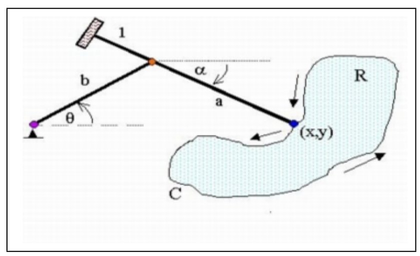


Sir George Gabriel Stokes
(1819–1903)

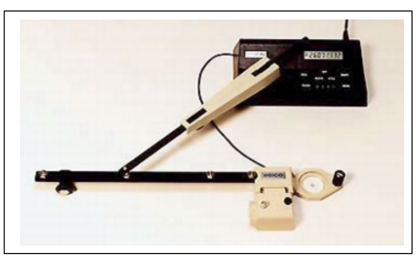
CHALLENGE: *Find a picture of George Green.*

Application: Planimeter

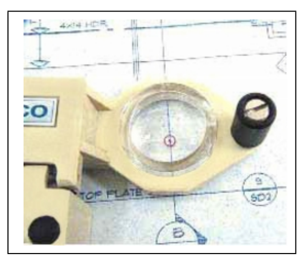
Lasico Planimeter ($\approx$ $150)



Theory of Operation



Lasico Digital Planimeter ($518)



Detail

A *planimeter* is an instrument that calculates an area by tracing the outline of a region in an architectural drawing.

6. Describe how Green's Theorem is used by a planimeter.