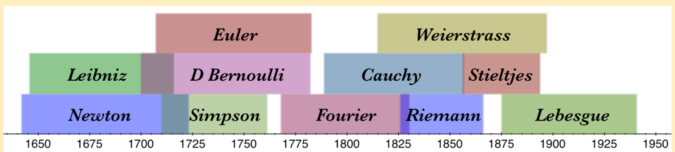
Introduction to Lebesgue Measure

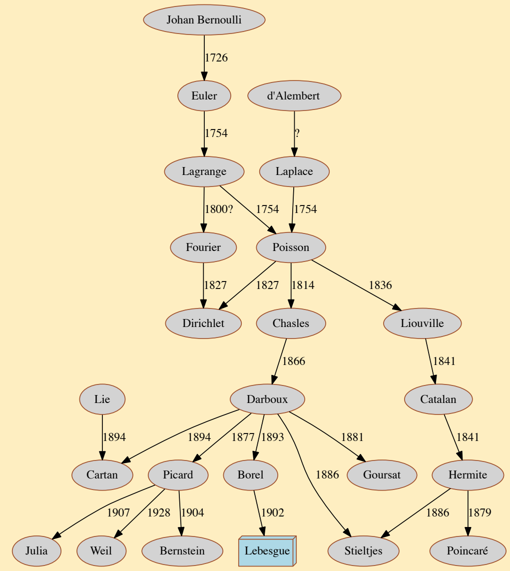
Prelude

There were two problems with calculus: there are functions where

- $f(x) \neq \int f'(x)\,dx$ (c.f., G-O #30)
- $f(x) \neq \dfrac{d}{dx}\left[\int f(x)\,dx\right]$

In his 1902 dissertation, *"Intégrale, longueur, aire,"* Lebesgue wrote, "It thus seems to be natural to search for a definition of the integral which makes integration the inverse operation of differentiation in as large a range as possible."





Henri Lebesgue's Mathematical Genealogy (partial)

Mine

What's in a Measure

Goals

THE BEST *measure* would be a real-valued set function μ that satisfies

1. $\mu(I) = \mathrm{length}(I)$ where I is an interval
2. μ is *translation invariant*: $\mu(x+E) = \mu(E)$ for any $x \in \mathbb{R}$
3. if $\{E_n\}$ is pairwise disjoint, then $\mu(\bigcup_n E_n) = \sum_n \mu(E_n)$
4. $\mathrm{dom}(\mu) = \mathcal{P}(\mathbb{R})$ (the power set of $\mathbb{R}$)

THE BAD NEWS:

$$\left\{\begin{array}{c}\textit{continuum hypothesis}\\ + \quad \textit{axiom choice}\end{array}\right\} \implies \text{1, 3, and 4 are incompatible}$$

THE PLAN:

- Give up on 4. (cf. *Vitali*)
- 1. and 2. are nonnegotiable
- Weaken 3., then reclaim it

Sigma Algebras

Definition

Sigma Algebra of Sets

Algebra: A collection of sets $\mathcal{A}$ is an *algebra* iff $\mathcal{A}$ is closed under unions and complements.

σ-Algebra: An algebra of sets $\mathcal{A}$ is a *σ-algebra* iff $\mathcal{A}$ is closed under countable unions.

Proposition

Let $\mathcal{A}$ be a nonempty algebra of sets of reals. Then

- *$\emptyset$ and $\mathbb{R} \in \mathcal{A}$. ($A \in \mathcal{A} \implies A^c \in \mathcal{A}$. Then $\mathbb{R} = A \cup A^c \in \mathcal{A}$. Then $\mathbb{R}^c \in \mathcal{A}$.)*
- *$\mathcal{A}$ is closed under intersection. ($A \cap B = [A^c \cup B^c]^c$ DeMorgan)*

Let $\mathcal{A}$ be a nonempty σ-algebra of sets of reals. Then

- *$\mathcal{A}$ is closed under countable intersections.*

Sigma Samples

Examples

1. $\mathcal{A} = \{\emptyset, \mathbb{R}\}$
2. $\mathcal{F} = \{F \subset \mathbb{R} : F$ is finite or F^c is finite$\}$
   1. $\mathcal{F}$ is an algebra, the *co-finite algebra*
   2. $\mathcal{F}$ is not a σ-algebra
      For each $r \in \mathbb{Q}$, the set $\{r\} \in \mathcal{F}$. But $\bigcup_{r\in\mathbb{Q}}\{r\} = \mathbb{Q} \notin \mathcal{F}$
3. Let $\mathcal{A} = \{\emptyset, [-1,1], (-\infty,-1) \cup (1,\infty), \mathbb{R}\}$. Is $\mathcal{A}$ an algebra?
4. Any intersection of σ-algebras is a σ-algebra
5. Let $\mathcal{B}(\mathbb{R})$ be the smallest σ-algebra containing all the open sets, the *Borel σ-algebra*.

Outer Measure

Definition (Lebesgue Outer Measure)

Let $E \subset \mathbb{R}$. Define the *Lebesgue Outer Measure* μ^* of E to be

$$\mu^*(E) = \inf_{E \subset \bigcup I_n} \sum_n \ell(I_n),$$

the infimum of the sums of the lengths of open interval covers of E.

Proposition (Monotonicity)

If $A \subseteq B$, then $\mu^*(A) \leq \mu^*(B)$.

Proposition

If I is an interval, then $\mu^*(I) = \ell(I)$.

Outer Measure of an Interval

Proof.

I. I is closed and bounded (compact). Then $I = [a, b]$.

1. For any $\varepsilon > 0$, $[a, b] \subset (a - \varepsilon, b + \varepsilon)$. So $\mu^*(I) \leq b - a + 2\varepsilon$. Since ε is arbitrary, $\mu^*(I) \leq b - a$.
2. Let $\{I_n\}$ cover $[a, b]$ with open intervals. There is a finite subcover for $[a, b]$. Order the subcover so that consecutive intervals overlap. Then

$$\sum_N \ell(I_k) = (b_1 - a_1) + (b_2 - a_2) + \cdots + (b_N - a_N)$$

Rearrange

$$\begin{aligned}\sum_N \ell(I_k) &= b_N - (a_N - b_{N-1}) - (a_{N-1} - b_{N-2}) - \cdots - (a_2 - b_1) - a_1 \\ &\geq b_N - a_1 > b - a\end{aligned}$$

Whence $\mu^*(I) = b - a$.

Outer Measure of an Interval, II

Proof (cont).

II. Let I be any bounded interval and $\varepsilon > 0$.

1. There is a closed interval $J \subset I$ so that $\ell(I) - \varepsilon < \ell(J)$. Then
$$\ell(I) - \varepsilon < \ell(J) = \mu^*(J) \leq \mu^*(I) \leq \mu^*(\bar{I}) = \ell(\bar{I}) = \ell(I)$$

III. Suppose I is infinite.

1. Then for each n, there is a closed interval $J \subset I$ s.t. $\ell(J) = n$
2. Thence $\mu^*(I) \geq n$ for all n.

Aha! $\mu^*(I) = \infty$

Proposition

$$\mu^*(\mathbb{Q}) = 0$$

Proof.

Order $\mathbb{Q}$ as $\{r_1, r_2, \dots\}$. The collection $\{I_n = (r_n - \varepsilon/2^n, r_n + \varepsilon/2^n)\}$ covers $\mathbb{Q}$ □

Countable Subadditivity

Theorem (μ^* is Countably Subadditive)

Let $\{E_n\}$ be a countable set sequence in $\mathbb{R}$. Then $\mu^*\left(\bigcup_n E_n\right) \leq \sum_n \mu^*(E_n)$

Proof.

I. If $\mu^*(E_n) = \infty$ for any n, then done.
II. Let $\varepsilon > 0$

1. For each n find a cover $\{I_{n,j}\}_{n\in\mathbb{N}}$ such that $\sum_{j\in\mathbb{N}} \ell(I_{n,j}) < \mu^*(E_n) + \frac{\varepsilon}{2^n}$
2. Then $\{I_{n,j}\}_{n,j\in\mathbb{N}}$ covers $E = \bigcup_n E_n$.
3. Whereupon
$$\mu^*(E) \leq \sum_{n,j\in\mathbb{N}} \ell(I_{n,j}) = \sum_{n\in\mathbb{N}} \left[\sum_{j\in\mathbb{N}} \ell(I_{n,j})\right]$$
$$< \sum_{n\in\mathbb{N}} \left[\mu^*(E_n) + \frac{\varepsilon}{2^n}\right] = \sum_{n\in\mathbb{N}} [\mu^*(E_n)] + \varepsilon$$

□

Open Holding & Lebesgue's Measure

Corollary

Given $E \subseteq \mathbb{R}$ and $\varepsilon > 0$, there is an open set $O \supseteq E$ s.t.

$$\mu^*(E) \le \mu^*(O) \le \mu^*(E) + \varepsilon$$

Definition (Carathéodory's Condition)

A set E is *Lebesgue measurable* iff for every (test) set A,

$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Let $\mathfrak{M}$ be the collection of all Lebesgue measurable sets.

Corollary

For any A and E,

$$\mu^*(A) = \mu^*\big((A \cap E) \cup (A \cap E^c)\big) \le \mu^*(A \cap E) + \mu^*(A \cap E^c)$$

Much Ado About Nothing

Theorem

If $\mu^*(E) = 0$, then $E \in \mathfrak{M}$; i.e., E is measurable.

Proof.

Given the previous corollary, we need only show that

$$\mu^*(A \cap E) + \mu^*(A \cap E^c) \le \mu^*(A)$$

1. Since $A \cap E \subset E$, then $\mu^*(A \cap E) \le \mu^*(E) = 0$.
2. Since $A \cap E^c \subset A$, then $\mu^*(A \cap E^c) \le \mu^*(A)$.

Whence $\mu^*(A \cap E) + \mu^*(A \cap E^c) \le 0 + \mu^*(A) = \mu^*(A)$. □

Corollary

$$\mu^*(\mathbb{Q}) = 0 \implies \mathbb{Q} \in \mathfrak{M}$$

Unions Work

Theorem

A finite union of measurable sets is measurable.

Proof.

Let E_1 and $E_2 \in \mathfrak{M}$. Let A be a test set.

1. Use $A \cap E_1^c$ as a test set for E_2 which is measurable. Thence
$$\mu^*(A \cap E_1^c) = \mu^*((A \cap E_1^c) \cap E_2) + \mu^*((A \cap E_1^c) \cap E_2^c)$$
2. Note $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^c)$. Whereupon
$$\begin{aligned}\mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1 \cup E_2)^c)&\\ &= \mu^*(A \cap (E_1 \cup E_2)) + \mu^*(A \cap (E_1^c \cap E_2^c))\\ &\le \left[\mu^*(A \cap E_1) + \mu^*(A \cap E_2 \cap E_1^c)\right] + \mu^*(A \cap E_1^c \cap E_2^c)\\ &\le \mu^*(A \cap E_1) + \left[\mu^*(A \cap E_1^c \cap E_2) + \mu^*(A \cap E_1^c \cap E_2^c)\right]\\ &= \mu^*(A \cap E_1) + \mu^*(A \cap E_1^c)\\ &= \mu^*(A)\end{aligned}$$

□

Countable Unions Work

Theorem

The countable union of measurable sets is measurable.

Proof.

Let $E_k \in \mathfrak{M}$ and $E = \bigcup_n E_n$. Choose a test set A.
We need to show $\mu^*(A \cap E) + \mu^*(A \cap E^c) \le \mu^*(A)$.

1. Set $F_n = \bigcup^n E_k$ and $F = \bigcup^\infty E_k = E$. Define $G_1 = E_1$, $G_2 = E_2 - E_1$, $\ldots$, $G_k = E_k - \bigcup^{k-1} E_j$, and $G = \bigcup G_k$. Then
(i) $G_i \cap G_j = \emptyset,\ (i \neq j)$ (ii) $F_n = \bigcup^n G_k$ (iii) $F = G = E$
2. Test F_n with A to obtain $\mu^*(A) = \mu^*(A \cap F_n) + \mu^*(A \cap F_n^c)$
3. Test G_n with $A \cap F_n$ to obtain
$$\begin{aligned}\mu^*(A \cap F_n) &= \mu^*((A \cap F_n) \cap G_n) + \mu^*((A \cap F_n) \cap G_n^c)\\ &= \mu^*(A \cap G_n) + \mu^*(A \cap F_{n-1})\end{aligned}$$

Countable Unions Work, II

Proof.

4. Iterate $\mu^*(A \cap F_n) = \mu^*(A \cap G_n) + \mu^*(A \cap F_{n-1})$ from 3 to have
$$\mu^*(A \cap F_n) = \sum_{k=1}^{n} \mu^*(A \cap G_k)$$
5. Since $F_n \subseteq F$, then $F^c \subseteq F_n^c$ for all n, then
$$\mu^*(A \cap F_n^c) \geq \mu^*(A \cap F^c)$$
6. Whence
$$\mu^*(A) \geq \sum_{k=1}^{n} \mu^*(A \cap G_k) + \mu^*(A \cap F^c)$$
The summation is increasing & bounded, so convergent.
7. However
$$\sum_{k=1}^{\infty} \mu^*(A \cap G_k) \geq \mu^*\left(\bigcup_{k=1}^{\infty}(A \cap G_k)\right) = \mu^*(A \cap F)$$

Aha! $\mu^*(A) \geq \mu^*(A \cap F) + \mu^*(A \cap F^c)$ □

Everything Works

Corollary

The collection of Lebesgue measurable sets $\mathfrak{M}$ is a σ-algebra.

Corollary

The Borel sets are measurable. (There are measurable, non-Borel sets.)
$$\mathcal{B}(\mathbb{R}) \subsetneqq \mathfrak{M} \subsetneqq \mathcal{P}(\mathbb{R})$$

Definition (Lebesgue Measure)

Lebesgue measure μ is μ^* restricted to $\mathfrak{M}$. So $\mu : \mathfrak{M} \to [0, \infty]$.

Definition (Almost Everywhere)

A property P holds *almost everywhere* (a.e.) iff $\mu\big(\{x : \neg P(x)\}\big) = 0$.