The Return of Additivity

Theorem

Let $\{E_n\}$ be a countable (finite or infinite) sequence of pairwise disjoint sets in $\mathfrak{M}$. Then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$

Proof.

I. n is finite.

1. For $n = 1$, ✓
2. $\left(\bigcup_{k=1}^{n} E_k\right) \cap E_n = E_n$ and $\left(\bigcup_{k=1}^{n} E_k\right) \cap E_n^c = \bigcup_{k=1}^{n-1} E_k$
3. $\mu\left(\bigcup_{k=1}^{n} E_k\right) = \mu\left(\left[\bigcup_{k=1}^{n} E_k\right] \cap E_n\right) + \mu\left(\left[\bigcup_{k=1}^{n} E_k\right] \cap E_n^c\right)$
$= \mu(E_n) + \mu\left(\bigcup_{k=1}^{n-1} E_k\right) = \mu(E_n) + \sum_{k=1}^{n-1} \mu(E_k) = \sum_{k=1}^{n} \mu(E_k)$

II. n is infinite.

1. $\bigcup_{k=1}^{n} E_k \subset \bigcup_{k=1}^{\infty} E_k \implies \mu\left(\bigcup_{k=1}^{n} E_k\right) = \sum_{k=1}^{n} \mu(E_k) \le \mu\left(\bigcup_{k=1}^{\infty} E_k\right)$
2. A bounded & increasing sum converges. Thus $\sum_{k=1}^{\infty} \mu(E_k) \le \mu\left(\bigcup_{k=1}^{\infty} E_k\right)$
3. Subadditivity finishes the proof. □

Adding an Example

Example

Set $E_n = \left(\frac{n-1}{n}, \frac{n}{n+1}\right)$ for $n = 1..\infty$.

1. The E_n are pairwise disjoint.
2. $\mu(E_n) = \ell(E_n) = \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)}$
3. $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = \sum_{n=1}^{\infty} \mu(E_n) = \sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \sum_{n=1}^{\infty} \left[\frac{1}{n} - \frac{1}{n+1}\right]$

Whence $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 1.$

NOTA BENE: $\bigcup_{n=1}^{\infty} E_n = (0,1) - \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$. Hence $\bigcup_{n=1}^{\infty} E_n = (0,1)$ *a.e.*

Matryoshka

Theorem

If $\{E_n\}$ is a seq of nested, measurable sets with $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n\to\infty} \mu(E_n)$$

Proof.

1. Set $E = \bigcap_{k=1}^{\infty} E_k$. Set $F_k = E_k - E_{k+1}$. The F_k are pairwise disjoint.
2. Since $\bigcup_{k=1}^{\infty} F_k = E_1 - E$, then $\mu(E_1 - E) = \sum_{k=1}^{\infty} \mu(F_k) = \sum_{k=1}^{\infty} \mu(E_k - E_{k+1})$.
3. If $A \supset B$, then $\mu(A - B) = \mu(A) - \mu(B)$. Apply to the formula above.
4. $\mu(E_1) - \mu(E) = \sum_{k=1}^{\infty} \mu(E_k) - \mu(E_{k+1}) = \mu(E_1) - \lim_{k\to\infty} \mu(E_k)$

Since $\mu(E_1)$ is finite, we're done. □

The Cantor Set

Cantor Sets[1]

I. Constructing C

1. Set $C_0 = [0, 1]$
2. Set $C_1 = C_0 - (\frac{1}{3}, \frac{2}{3})$
3. Set $C_2 = C_1 - (\frac{1}{3^2}, \frac{2}{3^2}) - (\frac{7}{3^2}, \frac{8}{3^2})$
4. Set $C_3 = C_2 - (\frac{1}{3^3}, \frac{2}{3^3}) - (\frac{7}{3^3}, \frac{8}{3^3}) - (\frac{19}{3^3}, \frac{20}{3^3}) - (\frac{25}{3^3}, \frac{26}{3^3})$
5. Let $C = \bigcap C_i$

II. Properties of C

1. $\mu(C_0) = 1$, $\mu(C_1) = 2/3$, $\mu(C_2) = 4/9$, $\mu(C_3) = 8/27, \ldots$ So $\mu(C_n) = \frac{2}{3}\mu(C_{n-1}) = \frac{2^n}{3^n}$ Whence $\mu(C) = 0$.
2. C is uncountable
3. C is perfect
4. C is nowhere dense
5. C is compact
6. C is totally disconnected
7. $(\forall i)\ \partial C_i \subset C$
8. $(\forall i)\ \frac{1}{4} \notin \partial C_i$, but $\frac{1}{4} \in C$

[1] Cantor gave the set in a footnote to show "perfect" $\not\subset$ "everywhere dense".

Not So Strange After All

Theorem

Let $E \subseteq \mathbb{R}$ and let $\varepsilon > 0$. TFAE:

1. *E is measurable*
2. *There is an open set $O \supset E$ s.t. $\mu^*(O - E) < \varepsilon$*
3. *There is a closed set $F \subset E$ s.t. $\mu^*(E - F) < \varepsilon$*

Proposition

Let S and T be measurable subsets of $\mathbb{R}$. Then

$$\mu(S \cup T) + \mu(S \cap T) = \mu(S) + \mu(T)$$

Functionally Measurable

Theorem (Measurability Conditions for Functions)

Let $f : D \to \mathbb{R}_\infty$ for some $D \in \mathfrak{M}$. TFAE

1. *For each $r \in \mathbb{R}$, the set $f^{-1}\big((r, \infty)\big)$ is measurable.*
2. *For each $r \in \mathbb{R}$, the set $f^{-1}\big([r, \infty)\big)$ is measurable.*
3. *For each $r \in \mathbb{R}$, the set $f^{-1}\big((-\infty, r)\big)$ is measurable.*
4. *For each $r \in \mathbb{R}$, the set $f^{-1}\big((-\infty, r]\big)$ is measurable.*

Proof.

$1 \Rightarrow 2$: $\{x \mid f(x) \geq r\} = \bigcap_n \{x \mid f(x) > r - 1/n\}$

$2 \Rightarrow 3$: $\{x \mid f(x) < r\} = D - \{x \mid f(x) \geq r\}$

$3 \Rightarrow 4$: $\{x \mid f(x) \leq r\} = \bigcap_n \{x \mid f(x) < r + 1/n\}$

$4 \Rightarrow 1$: $\{x \mid f(x) > r\} = D - \{x \mid f(x) \leq r\}$ □

The Measurably Functional

Corollary

If f satisfies any measurability condition, then $\{x \mid f(x) = r\}$ is measurable for each r.

Definition (Measurable Function)

If a function $f\colon D \to \mathbb{R}_\infty$ has measurable domain D and satisfies any of the measurability conditions, then f is *measurable*.

Definition

Step function: $\phi\colon [a,b] \to \mathbb{R}_\infty$ is a *step function* if there is a partition $a = x_0 < x_1 < \cdots < x_n = b$ s.t. ϕ is constant on each interval $I_k = (x_{k-1}, x_k)$, then

$$\phi(x) = \sum_{k=1}^{n} a_k \chi_{I_k}(x)$$

Simple function: A function ψ with range $\{a_1, a_2, \ldots, a_n\}$ where each set $\psi^{-1}(a_k)$ is measurable is a *simple function*.

Simply Stepping

Proposition

Step functions and simple functions are measurable

Theorem (Algebra of Measurable Functions)

Let f and g be measurable on a common domain D, and let $c \in \mathbb{R}$. Then

1. $f + c$
2. $c \cdot f$
3. $f \pm g$
4. f^2
5. $f \cdot g$

are all measurable.

Proof.

- ✓ □

Sequencing

Theorem

Let $\{f_n\}$ be a sequence of measurable functions on a common domain D. Then

1. $\sup\{f_1, \ldots, f_n\}$
2. $\inf\{f_1, \ldots, f_n\}$
3. $\sup_{n\to\infty} f_n$
4. $\inf_{n\to\infty} f_n$
5. $\limsup_{n\to\infty} f_n$
6. $\liminf_{n\to\infty} f_n$

are all measurable.

Proof.

1. Set $f = \{f_1, \ldots, f_n\}$. Then $\{f(x) > r\} = \bigcup_{k=1}^{n} \{f_k(x) > r\}$.
3. Set $F = \sup_n f_n$. Then $\{F(x) > r\} = \bigcup_{k=1}^{\infty} \{f_k(x) > r\}$.
5. Set $\Phi = \limsup_n f_n$. Then $\limsup_{n\to\infty} f_n = \inf_n \left[\sup_{k\geq n} f_k\right]$

□

Zeroing

Theorem

If f is measurable and $f = g$ a.e., then g is measurable.

Definition (Converence Almost Everywhere)

A sequence $\{f_n\}$ *converges to f almost everywhere*, written as $f_n \to f$ a.e., iff $\mu\Big(\{x : f_n(x) \not\to f(x)\}\Big) = 0$.

Theorem

Let $f : [a, b] \to \mathbb{R}$. Then f is measurable iff there is a seq. of simple functions $\{\psi_n\}$ converging to f a.e.

A Simple Proof

Proof.

($\Rightarrow$) Wolog $f \geq 0$.

1. Define $A_{n,k} = \left\{x \,\middle|\, \frac{k-1}{2^n} \leq f(x) < \frac{k}{2^n}\right\}$ for $k = 1..(n \cdot 2^n)$ and
$$A_{0,n} = [a,b] - \bigcup_{k=1}^{n2^n} A_{n,k}$$
2. Set $\psi_n(x) = n\chi_{A_{0,n}}(x) + \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \cdot \chi_{A_{n,k}}(x)$
3. Then
    1. $\psi_1 \leq \psi_2 \leq \cdots$
    2. If $0 \leq f(x) \leq n$, then $|f - \psi_n| < 2^{-n}$
    3. $\lim_n \psi = f$ a.e.

($\Leftarrow$) ✓ □

Maple Example

Integration

We began by looking at two examples of integration problems.

- The Riemann integral over $[0,1]$ of a function with infinitely many discontinuities didn't exist even though the points of discontinuity formed a set of measure zero.
  (The points of discontinuity formed a dense set in $[0,1]$.)
- The limit of a sequence of Riemann integrable functions did not equal the integral of the limit function of the sequence.
  (Each function had area $^1/_2$, but the limit of the sequence was the zero function.)

We will look at Riemann integration, then Riemann-Stieltjes integration, and last, develop the Lebesgue integral.

There are many other types of integrals: Darboux, Denjoy, Gauge, Perron, etc. See the list given in the "See also" section of *Integrals* on Mathworld.