Intro to Lebesgue Measure

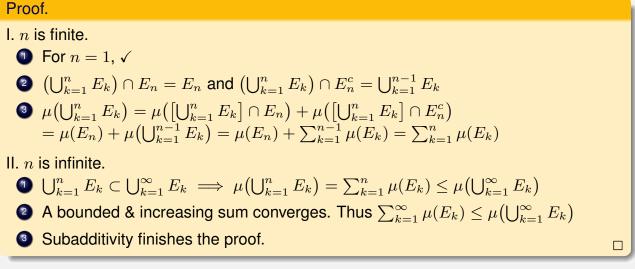
Lebesque Measure

The Return of Additivity

Theorem

Let $\{E_n\}$ be a countable (finite or infinite) sequence of pairwise disjoint sets in \mathfrak{M} . Then

$$\mu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \mu(E_k)$$



WmCB (BauldryWC@appstate.edu)

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Adding an Example

Example Set $E_n = \left(\frac{n-1}{n}, \frac{n}{n+1}\right)$ for $n = 1..\infty$. • The E_n are pairwise disjoint. 2 $\mu(E_n) = \ell(E_n) = \frac{n}{n+1} - \frac{n-1}{n} = \frac{1}{n(n+1)}$ Whence $\mu\left(\bigcup_{n=1}^{\infty} E_n\right) = 1.$ NOTA BENE: $\bigcup_{n=0}^{\infty} E_n = (0,1) - \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \dots\}$. Hence $\bigcup_{n=0}^{\infty} E_n = (0,1)$ a.e.

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Matryoshka

Theorem

If $\{E_n\}$ is a seq of nested, measurable sets with $\mu(E_1) < \infty$, then

$$\mu\left(\bigcap_{n=1}^{\infty} E_n\right) = \lim_{n \to \infty} \mu(E_n)$$

Proof.

Set
$$E = \bigcap_{k=1}^{\infty} E_k$$
. Set $F_k = E_k - E_{k+1}$. The F_k are pairwise disjoint.
Since $\bigcup_{k=1}^{\infty} F_k = E_1 - E$, then $\mu(E_1 - E) = \sum_{k=1}^{\infty} \mu(F_k) = \sum_{k=1}^{\infty} \mu(E_k - E_{k+1})$.
If $A \supset B$, then $\mu(A - B) = \mu(A) - \mu(B)$. Apply to the formula above.
 $\mu(E_1) - \mu(E) = \sum_{k=1}^{\infty} \mu(E_k) - \mu(E_{k+1}) = \mu(E_1) - \lim_{k \to \infty} \mu(E_k)$
Since $\mu(E_1)$ is finite, we're done.

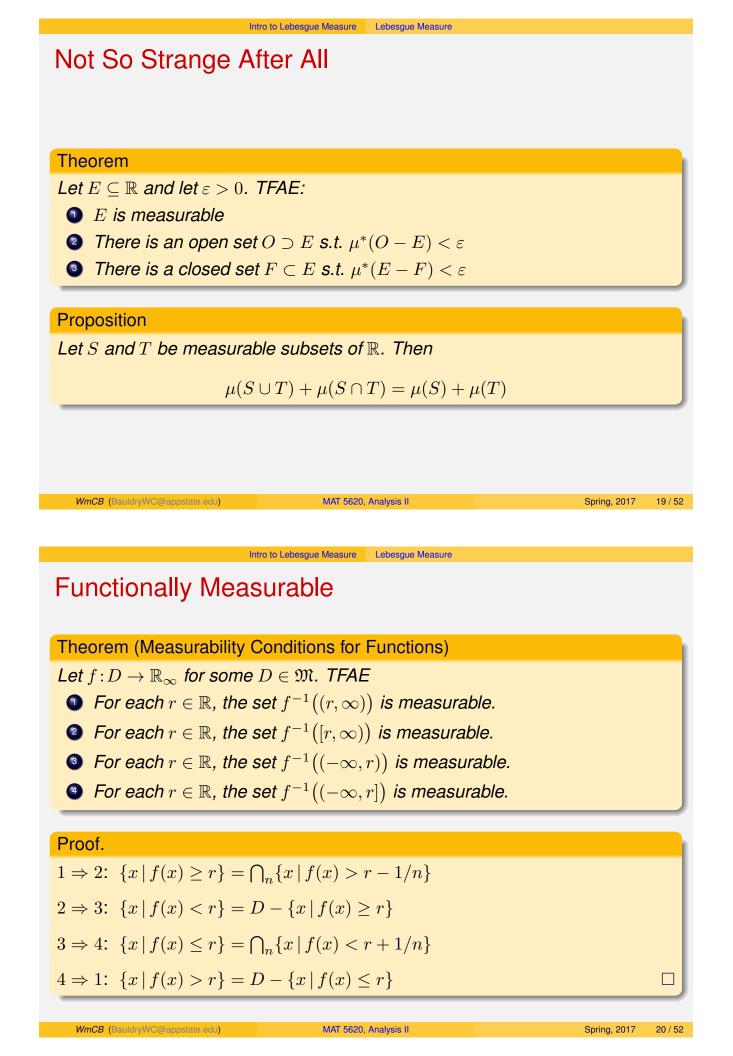
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WmCB (BauldryWC@appstate.edu)
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The Measurably Functional

Corollary

If *f* satisfies any measurability condition, then $\{x \mid f(x) = r\}$ is measurable for each *r*.

Definition (Measurable Function)

If a function $f: D \to \mathbb{R}_{\infty}$ has measurable domain D and satisfies any of the measurability conditions, then f is *measurable*.

Definition

Step function: $\phi:[a,b] \to \mathbb{R}_{\infty}$ is a *step function* if there is a partition $a = x_0$ $\langle x_1 < \cdots < x_n = b$ s.t. ϕ is constant on each interval $I_k = (x_{k-1}, x_k)$, then

$$\phi(x) = \sum_{k=1}^{n} a_k \chi_{I_k}(x)$$

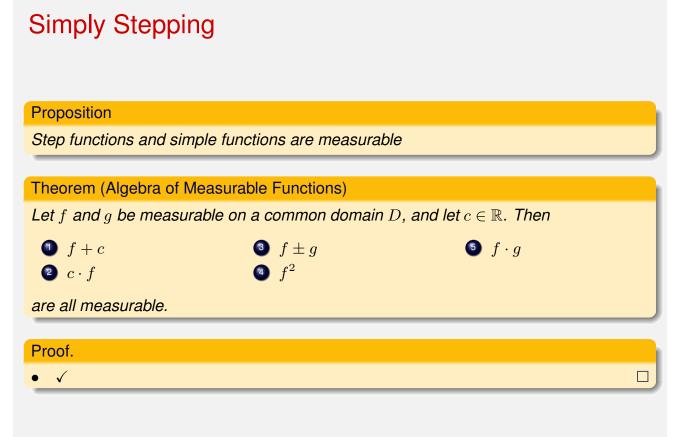
Simple function: A function ψ with range $\{a_1, a_2, \ldots, a_n\}$ where each set $\psi^{-1}(a_k)$ is measurable is a *simple function*.

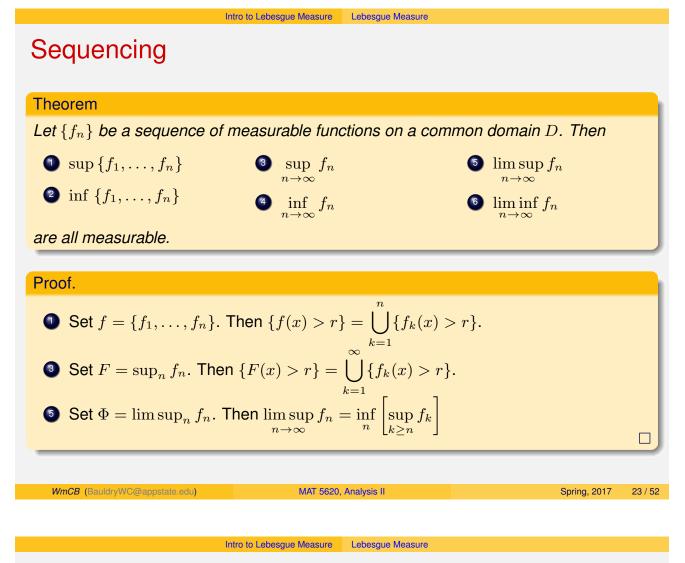
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Zeroing

Theorem

If f is measurable and f = g a.e., then g is measurable.

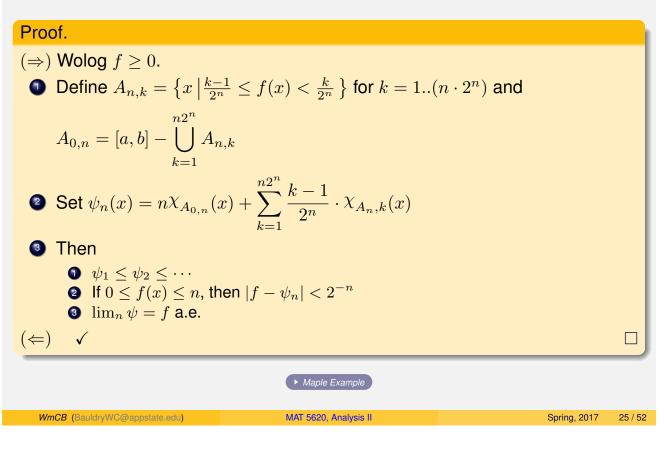
Definition (Converence Almost Everywhere)

A sequence $\{f_n\}$ converges to f almost everywhere, written as $f_n \to f$ a.e., iff $\mu(\{x: f_n(x) \not\to f(x)\}) = 0.$

Theorem

Let $f:[a,b] \to \mathbb{R}$. Then f is measurable iff there is a seq. of simple functions $\{\psi_n\}$ converging to f a.e.

A Simple Proof



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Integration

We began by looking at two examples of integration problems.

• The Riemann integral over [0, 1] of a function with infinitely many discontinuities didn't exist even though the points of discontinuity formed a set of measure zero.

(The points of discontinuity formed a dense set in [0, 1].)

 The limit of a sequence of Riemann integrable functions did not equal the integral of the limit function of the sequence. (Each function had area ¹/₂, but the limit of the sequence was the zero function.)

We will look at Riemann integration, then Riemann-Stieltjes integration, and last, develop the Lebesgue integral.

There are many other types of integrals: Darboux, Denjoy, Gauge, Perron, etc. See the list given in the "See also" section of *Integrals* on Mathworld.