①

Line Int Eg. 1

$$C(t) = \langle t, 2t^2, t^3 \rangle \text{ on } [0,1]$$

$$f(\bar{x}) = \langle x_1, x_1 x_3 + 1, x_1 + x_2 x_3 \rangle$$

$$\int_{\mathcal{C}} f(\bar{x}) \cdot dx = \int_C f(\bar{x})\, x' \cdot dt$$

$$= \int_0^1 \langle t, t^4+1, t+2t^5 \rangle \cdot \langle 1, 4t, 3t^2 \rangle dt$$

$$= \int_0^1 \left( t + (t^4+1)(4t) + (t+2t^5)(3t^2) \right) dt$$

$$= \int_0^1 (6t^7 + 4t^5 + 3t^3 + 5t)\, dt = \frac{14}{3}$$

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Open, connected sets ("path conn")

- Draw egs

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FToC on $\mathbb{R}^1$ $\int_a^b \phi'(t) = \phi(b) - \phi(a)$ $\phi'$: cont on $[a,b]$

(1.5) Eg 2

$$F(x,y) = \begin{bmatrix} \sqrt{x^2+y^2} \\ \sqrt{1-x^2} \end{bmatrix}$$

$$\alpha = \begin{bmatrix} \cos(t) \\ \sin(t) \end{bmatrix} \quad t \in [0, \pi]$$

$$\beta = \begin{bmatrix} -\sin(2\tau) \\ \cancel{\cos(2t)} \end{bmatrix} \quad \tau \in \left[-\frac{\pi}{4}, \frac{\pi}{4}\right] \quad \leftarrow \text{graph}$$

$$F(\alpha(t)) = \begin{bmatrix} 1 \\ \sin(t) \end{bmatrix} \quad t \in [0, \pi]$$

$$\int_\alpha F \cdot d\alpha = \int_0^\pi \begin{bmatrix} 1 \\ \sin(t) \end{bmatrix} \cdot \begin{bmatrix} -\sin(t) \\ \cos(t) \end{bmatrix} dt = -2$$

$$\int_\beta F \cdot d\beta = \int_{-\pi/4}^{\pi/4} \begin{bmatrix} 1 \\ \cos(2t) \end{bmatrix} \cdot \begin{bmatrix} -\cos(2t) \cdot 2 \\ -\sin(2t) \cdot 2 \end{bmatrix} dt = -2$$

!

back to open, conn set

②

Thm 10.3 2nd FToC for Line Integrals

Let $\phi$ : diffy scalar field

grad: $\nabla \phi$ is continuous on open connected set $S \subseteq \mathbb{R}^n$

let $\bar{a}, \bar{b} \in S$

let $\alpha$ be any pw smooth path from $\bar{a}$ to $\bar{b}$ in $S$
$: [a,b] \to S$

Then $\int_a^b \nabla \phi \cdot d\alpha = \phi(b) - \phi(a)$

pf: wolog $\alpha$ is smooth on $[a,b]$

Then $\int_\alpha \nabla \phi \cdot d\alpha = \int_a^b \nabla \phi(\alpha) \cdot \alpha'(t)\, dt$

Chain Rule $\Rightarrow$ $\nabla \phi(\alpha) \cdot \alpha'(t) = g'(t)$ for

$g = \phi \circ \alpha$

Then $g'$ is cont on $(a,b)$ since $\alpha$ : smooth & $\nabla \phi$ : cont on $S$

$$\therefore \int_\alpha \nabla \phi \cdot d\alpha = \int_a^b g'(t)\, dt \underset{\substack{\text{FToC} \\ \mathbb{R}^1}}{=} g(b) - g(a)$$

$$= \phi(\bar{b}) - \phi(\bar{a})$$

$\boxtimes$

$\alpha$ : pw-smooth

$$\int_\alpha \nabla \phi \cdot d\alpha = \sum_k \int_{\alpha_k} \nabla \phi \cdot d\alpha_k$$

$$= \sum_k \phi(\alpha(t_{k+1})) - \phi(\alpha(t_k))$$

$$= \phi(\bar{b}) - \phi(\bar{a})$$

③ ~~Since~~ Then $\phi(b) - \phi(a)$ is indep of path on subset where $\nabla\phi$ is cont.

FToC $\mathbb{R}^1$ #1

$$\phi(x) = \int_a^x f(t)\,dt \Rightarrow \phi'(x) = f(x)$$

Th$^m$ 10.4 FToC 1 for line Int.

Let $f$: vector field cont on open, conn $S \subseteq \mathbb{R}^n$

Let $\int_{\bar a}^{\bar b} f$ be indep of path in $S$

Let $\bar a \in S$

Define $\phi(\bar x) = \int_{\bar a}^{\bar x} f \cdot d\alpha$ for any pw-smooth $\alpha \subseteq S$

Then $\nabla\phi$ exists & $\nabla\phi(\bar x) = f(\bar x)$ $\forall \bar x \in S$

Pf. Since $\nabla\phi = \langle \frac{\partial}{\partial x_k}\phi \rangle$ only NTS $\frac{\partial}{\partial x_k}\phi = f_k$

Find a ball $B(\bar x; r)^{\subseteq S}$ ~~s.t.~~ then ~~[illegible]~~
for any unit vector $\bar y$ we have $\bar x + h\bar y \in B$ when $|h| < r$

Consider
$$\frac{\phi(\bar x + h\bar y) - \phi(\bar x)}{h} = \frac{1}{h} \cdot \int_{\bar x}^{\bar x + h\bar y} f \cdot d\alpha$$

for $\alpha$: segment $\bar x$ to $\bar x + h\bar y$ $= \alpha(t) = \bar x + th\bar y,\ t \in [0,1]$

$$DQ = \frac{1}{h}\int_0^1 f(\bar x + th\bar y) \cdot y \cdot h\,dt$$

(4)

$$DQ = \int_0^1 f(\bar{x} + th\bar{y}) \cdot \bar{y}\, dt$$

Let $\bar{y} = \bar{e}_k$ ; then

$$f(\bar{x} + th\bar{y}) \cdot \bar{y} = f(\bar{x} + th\bar{e}_k) \cdot \bar{e}_k$$
$$= f_k(\bar{x} + th\bar{e}_k)$$

Set $u = t \cdot h \to du = h.dt$

$$DQ = \int_0^h f_k(\bar{x} + u\bar{e}_k) \cdot \frac{1}{h}\, du$$

$$= \frac{1}{h} \int_0^h f_k(\bar{x} + u\bar{e}_k)\, du$$

$$= \frac{g(h) - g(0)}{h} \quad (*) \quad \text{for } g(t) = \int_0^t f_k(\bar{x} + u\bar{e}_k)\, du$$

$f_k$ : cont $\to$ $g$ : cont

$\therefore \; g'(t) = f_k(\bar{x} + t\bar{e}_k)$ & $g'(0) = f_k(\bar{x})$

let $h \to 0$ in $(*)$ : $\therefore \lim_{h \to 0} DQ = g'(0) = f_k(\bar{x})$

i.e. $\dfrac{\partial}{\partial x_k} \phi(\bar{x}) = f_k(\bar{x})$

Done.