The Greening of Multiple Integrals

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Double Integrals

Theorem (11.5 Iterated Integrals)

Let f be bounded and integrable on $S = [a, b] \times [c, d]$. Assume the integrals $A(y) = \int_a^b f(x, y)\,dx$ exist for each fixed y. Assume $\int_c^d A(y)\,dy$ exists. Then

$$\iint_S f\,dx\,dy = \int_c^d \left[\int_a^b f(x, y)\,dx \right] dy$$

Examples

1. $\displaystyle\iint_S x\ \sin(y)\,dx\,dy$ with $S = [0, 2] \times [0, \pi/2]$
2. $\displaystyle\iint_R \left[\frac{x}{y} + \frac{y}{x} \right] dx\,dy$ with $R = [1, 2] \times [1, 2]$
3. $\displaystyle\iint_S xy\,e^{x^2 y}\,dA$ with $S = [0, 1] \times [0, 2]$
4. $\displaystyle\iint_R \frac{xy}{x^2 + 1}\,dA$ with $R = [0, 1] \times [1, 3]$

Multiple Integrals

Theorem (11.6 Integrability of Continuous Functions - Fubini)

If $f : S \to \mathbb{R}$ is continuous on $S = [a, b] \times [c, d]$, then f is integrable on S. And

$$\iint_S f\, dA = \int_c^d \left[\int_a^b f(x, y)\, dx \right] dy = \int_a^b \left[\int_c^d f(x, y)\, dy \right] dx$$

Definition (Content Zero)

A bounded set A has *content zero* iff A can be covered by a set of rectangles with total area $< \varepsilon$.

Theorem (11.7 Integrability of Almost Continuous Functions)

If $f : S \to \mathbb{R}$ is bounded on $S = [a, b] \times [c, d]$ and continuous on S except on a set on content zero, then $\iint_S f\, dA$ exists.

Multiple Integrals

Definition (Integrals over Nonrectangular Regions)

Let S be a bounded region enclosed in a rectangle R. Let $f : S \to \mathbb{R}$ be bounded. Extend to f to R by

$$\bar{f}(x,y) = \begin{cases} f(x,y) & x \in S \\ 0 & x \in R - S \end{cases}$$

Define

$$\iint_S f\,dA = \iint_R \bar{f}\,dA$$

Theorem

The graph $\{(t, \phi(t))\}$ of a continuous function ϕ has content zero.

Theorem (11.9 Nonrectangular Regions)

Let $f : S \to \mathbb{R}$ be continuous on $S = \left\{a \le x \le b \text{ and } \phi_1(x) \le y \le \phi_2(x)\right\}$. Then

$$\iint_S f\,dA = \int_a^b \left[\int_{\phi_1(x)}^{\phi_2(x)} f(x,y)\,dy\right] dx$$

Multiple Integrals

Examples

1. $\displaystyle\int_0^1 \int_0^x (3 - x - y)\, dy\, dx$

   (*Do 2 ways*)

2. $\displaystyle\int_0^1 \int_{\sqrt{y}}^1 dx\, dy$

   (*Do 2 ways*)

3. $\displaystyle\int_0^1 \int_0^x \frac{\sin(x)}{x}\, dy\, dx$

   (*Try to do 2 ways*)

4. $\displaystyle\int_{-\pi/3}^{\pi/3} \int_0^{\sec(t)} 3\cos(t)\, du\, dt$

5. $\displaystyle\int_0^1 \int_{-\sqrt{1-y^2}}^{\sqrt{1-y^2}} 3y\, dx\, dy$

6. Find the volume of the region that lies under the paraboloid $z = x^2 + y^2$ and above the triangle enclosed by the lines $x = 0$, $y = x$, and $x + y = 2$ in the xy-plane.

§11.19 Green's Theorem in the Plane

Theorem (Green's Theorem [v1.0])

Let P & Q be scalar fields continuously differentiable on an open set $S \subset \mathbb{R}^2$ that contains a piecewise-smooth Jordan curve C and R, the interior of C. Then

$$\oint_C P\,dx + Q\,dy = \iint_R \left[\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right] dx\,dy$$

Note

$$\oint_C Q\,dy = \iint_R \frac{\partial Q}{\partial x}\,dx\,dy \quad \textit{and} \quad \oint_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dx\,dy$$

Example (Planimeter)

$$\text{Area}(R) = \oint_C (-\tfrac{1}{2}y)\,dx + (\tfrac{1}{2}x)\,dy = \iint_R dA$$

▸ Wikipedia ▸ Acme 'Planimeter' ▸ Digital Planimeter ▸ History ▸ Build one

Green's Theorem - Simplified Proof

Proof. [One direction of a special case.]

Suppose R is an "x-simple" region and let $C = \partial R$; i.e., for some continuous ϕ_1 & ϕ_2, we have $R \cup C = \{(a \le x \le b) \wedge (\phi_1(x) \le y \le \phi_2(x))\}$. Then

$$\oint_C P\,dx = \int_a^b P(x, \phi_1(x))\,dx - \int_a^b P(x, \phi_2(x))\,dx \tag{1}$$

By the FToC

$$\begin{aligned} \iint_R \frac{\partial P}{\partial y}\,dA &= \int_a^b \int_{\phi_1(x)}^{\phi_2(x)} \frac{\partial P}{\partial y}\,dy\,dx \\ &= \int_a^b P(x, \phi_2(x))\,dx - \int_a^b P(x, \phi_1(x))\,dx \end{aligned} \tag{2}$$

Combine (1) and (2) to get

$$\oint_C P\,dx = -\iint_R \frac{\partial P}{\partial y}\,dA$$

□

Green's Theorem - Example

Example

Find the area of the unit circle using Green's Theorem.