Outline

§10.2

Definition 1 (Path)**.** $\alpha : [a, b] \to \mathbb{R}^n$

- *Continuous Path:* α *is continuous*
- *Smooth Path:* α *is continuously differentiable*
- *Piecewise Smooth Path:* $[a, b] = \cup[a_k, b_k]$ *and* α *is smooth on each* $[a_k, b_k]$

Eg. spacecurve

Definition 2 (Line Integral)**.**

- α*: piecewise smooth path*
- $f : \mathbb{R}^n \to \mathbb{R}^m$*: vector field bounded on* $\alpha([a, b])$*:* graph of α

$$\int f \bullet d\alpha = \int_a^b f\big(\alpha(t)\big) \bullet \alpha'(t)\, dt$$

($a \bullet b$ is the *inner/dot product*)

§10.3
If $\vec{a} = \alpha(a) = \vec{b} = \alpha(b)$, then the path is **closed**.
For a closed, piecewise smooth path α, write $\oint f \bullet d\alpha$

For $f = (f_1, \dots, f_n)$ and $\alpha = (\alpha_1, \dots, \alpha_n)$:

$$\int f \bullet d\alpha = \sum_{k=1}^{n} \int_a^b f_k\big(\alpha(t)\big) \cdot \alpha_k'(t)\, dt$$

For $\mathbb{R}^2$ with $\alpha(t) = (\alpha_1(t), \alpha_2(t))$: let C be the path α and $x = \alpha_1(t)$, $y = \alpha_2(t)$. Then

$$\int_C f \bullet d\alpha = \int_C f_1(x, y)\, dx + f_2(x, y)\, dy$$

E.g. Evaluate $\oint_C f \bullet d\alpha$ where $f(x, y) = (y^2, x)$ and

- C is given by C_1 = the line segment from $(-5, -3)$ to $(0, 2)$
- C is given by C_2 = the arc of parabola $x + y^2 = 4$ from $(-5, -3)$ to $(0, 2)$

1. $C_1 : \alpha(t) = [-5, -3] \cdot (1 - t) + [0, 2] \cdot t$ or $x = 5t - 5$ and $y = 5t - 3$ for $t \in [0, 1]$. So

$$\int_{C_1} y^2\, dx + x\, dy = \int_0^1 (5t - 3)^2\, 5\, dt + (5t - 5)\, 5\, dt = -\frac{5}{6}$$

2. Evaluate $\int_{C_1} f(\alpha) \bullet \alpha'\, dt$

3. $C_2 : \alpha(t) = [4 - y^2, y]$ for $y \in [-3, 2]$. So

$$\int_{C_2} y^2\, dx + x\, dy = \int_{-3}^{2} y^2\, (-2y)\, dy + (4 - y^2)\, dy = 40\, \frac{5}{6}$$

§10.4
Linearity

1. $\int cf \bullet d\alpha$
2. $\int (f+g) \bullet d\alpha$

Additivity

1. $\int_{C_1} f \bullet d\alpha + \int_{C_2} f \bullet d\alpha$

Definition 3. *α and β are* equivalent piecewise smooth paths *iff* . . .

Theorem 1 (Change of Parameter)**.** *α and β: equivalent piecewise smooth paths* $\Longrightarrow$

$$\int_C f \bullet d\alpha = \int_C f \bullet d\beta$$

E.g. #12b p 328. $\oint_C y\,dx + z\,dy + x\,dz$ for C is $\{x^2+y^2=1\} \cap \{z=xy\}$
$f(x,y,z) = (y,z,x)$ and $\alpha(x,y,z) = (x,y,z)$