Vector Calculus

Vector Space Axioms

A set $\mathcal{V} = \{\vec{v}\}$ with addition $+$ and scalar multiplication $\cdot$ with scalars from a field F is a *vector space over* F when

1. $\langle \mathcal{V}, + \rangle$ is an Abelian group.
2. - scalar multiplication distributes over vector addition
   - scalar addition distributes over scalar multiplication
   - multiplication of scalars 'associates' with scalar multiplication

Recall:

- The *norm* (magnitude) of a vector $\vec{u}$ is $\|\vec{u}\| = \sqrt{\sum u_i^2}$
- The *direction vector* of $\vec{u}$ is $(1/\|\vec{u}\|) \cdot \vec{u}$

Definition (Dot Product in $\mathbb{R}^n$ over $\mathbb{R}$)

Dot Product $\quad \vec{u} \cdot \vec{v} = \sum u_i \cdot v_i = \|\vec{u}\| \, \|\vec{v}\| \, \cos(\angle \overline{uv})$

Dot Product

Proposition (Dot Product Properties)

Let $\vec{u}$ and $\vec{v}$ be in $\mathbb{R}^n$. Then

1. $\angle \overline{uv} = \cos^{-1}\left[\dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \, \|\vec{v}\|}\right]$ *angle between vectors*
2. $|\vec{u} \cdot \vec{v}| \leq \|\vec{u}\| \, \|\vec{v}\|$ *Cauchy-Bunyakovsky-Schwarz inequality*
3. $\|\vec{u} + \vec{v}\| \leq \|\vec{u}\| + \|\vec{v}\|$ *Triangle inequality; (cf. Minkowski's inequality)*
4. $\operatorname{proj}_{\vec{v}}(\vec{u}) = \dfrac{\vec{u} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \, \vec{v}$ *(orthogonal) projection of $\vec{u}$ onto $\vec{v}$*