Constructing a Nonmeasurable Set

Let $X = [0, 1)$ and define the operation $\oplus : X \to X$ by addition modulo 1. (Note: X is then equivalent to the unit circle via $t \mapsto e^{2\pi i t}$.) Let $A \oplus x = \{a \oplus x \mid a \in A\}$.

Define the relation $\sim$ by

$$x \sim y \text{ if and only if there is a rational } r \text{ such that } |x - y| = r$$

Exercises.

1. Show that for each rational r, we have $r \sim 0$, and so all rationals are equivalent under $\sim$.
2. Prove that $\sim$ is an equivalence relation on X.
3. Find $[0]$.

Consider $x_1 = \pi/10$ and $x_2 = \pi/30$. Since $x_1 - x_2 = \pi/15 \notin \mathbb{Q}$, then $[x_1] \neq [x_2]$. Now consider $x_3 = (\pi + 5)/10$. Since $x_1 - x_3 = 1/2 \in \mathbb{Q}$, we have that $[x_1] = [x_3]$.

Since $\sim$ is an equivalence relation, it partitions X. Choose a representative h from each equivalence class in the partition of X. (*Axiom of Choice*!) Gather these elements to form the set H. Consider the collection of these sets $\mathcal{H} = \{H \oplus r\}$ where r ranges over the rationals in X.

Exercises.

4. Determine whether H is countable or uncountable.
5. Verify that $\mathcal{H}$ is a pairwise-disjoint family; i.e., $(H \oplus r_1) \cap (H \oplus r_2) = \emptyset$ for $r_1 \neq r2$.
6. Prove that $X = \displaystyle\bigcup_{r \in \mathbb{Q} \cap X} (H + r)$.

Since Lebesgue measure is translation invariant, $\mu(H \oplus r) = \mu(H)$ for all $r \in \mathbb{Q} \cap X$. Assume that H is Lebesgue measurable with $\mu(H) = \lambda$. Then, since $\mathcal{H}$ is a countable family of disjoint sets,

$$1 = \mu(X) = \mu\left(\bigcup_{r \in \mathbb{Q} \cap X} (H + r)\right) = \sum_{r \in \mathbb{Q} \cap X} \lambda$$

We have a contradiction: If $\lambda = 0$, then $1 = 0$. Otherwise, if $\lambda > 0$, then $1 = \infty$. Thus H cannot be Lebesgue measurable.

This construction is due to Giuseppe Vitali. (See also *Vitali covering*.)