MAT 5620. Analysis II.
Notes on Measure Theory.

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Henri Léon Lebesgue
(1875–1941)

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"Riemann, We Have a Problem."

There are problems with Riemann integration.

1. Define Dirichlet's function (1829) $D(x) = \begin{cases} 1 & \text{if } x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{cases}$.

   Then $\int_{[0,1]} D(x)\,dx$ does not exist.

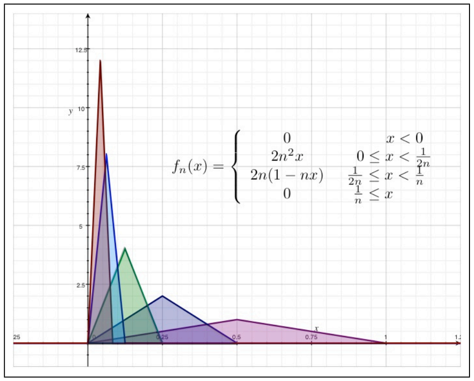
2. Set $f_n(x) = \begin{cases} 2n^2 x & 0 \le x < \frac{1}{2n} \\ 2n(1-nx) & \frac{1}{2n} \le x < \frac{1}{n} \\ 0 & \text{otherwise} \end{cases}$. Then

$$\int_{[0,1]} \lim_{n\to\infty} f_n(x)\,dx \neq \lim_{n\to\infty} \int_{[0,1]} f_n(x)\,dx.$$

Enter Henri Lebesgue in 1902.

A Bad Sequence of Functions

Example



- Find $\int f_n$, $\lim_n \int f_n$, $\lim_n f_n$, and $\int \lim_n f_n$.

Toward a Unit of Measure

Definition

The *length of an interval* in $\mathbb{R}^1$ is the difference of the endpoints and is given by $\ell([a, b]) = b - a$.

Goal: To have a set-function $m : \mathfrak{M} \to \mathbb{R}$ that "measures" the "size" of a set where m ideally satisfies:

1. $\mathfrak{M} = \mathcal{P}(\mathbb{R})$; *id est*, every set can be measured.
2. For every interval I, open or closed or not, $m(I) = \ell(I)$.
3. If the sequence $\{E_n\}$ is disjoint, then $m(\bigcup E_n) = \sum m(E_n)$.
4. m is *translation invariant;* i.e., $m(E + x) = m(E)$ for every E and any $x \in \mathbb{R}$.

Unfortunately, this is impossible.[1] We give up the first and allow sets not to be in the class of measurable sets, $\mathfrak{M} \subset \mathcal{P}(\mathbb{R})$.

[1] Even the first 3 are impossible assuming the *continuum hypothesis*.

σ-Algebra of Sets

Definition

Let $\mathcal{A}$ be a collection of sets. Then $\mathcal{A}$ is an *algebra of sets* or a *Boolean algebra* iff

- if $A \in \mathcal{A}$, then $A^c \in \mathcal{A}$,
- if $A, B \in \mathcal{A}$, then $A \cup B \in \mathcal{A}$.

De Morgan's laws imply that if $A, B \in \mathcal{A}$, then $A \cap B \in \mathcal{A}$. Then we also have $\emptyset \in \mathcal{A}$ and $X \in \mathcal{A}$.

Definition

Let $\mathcal{A}$ be an algebra of sets. Then $\mathcal{A}$ is a σ*-algebra* of sets or *Borel field* iff for every countable sequence $\{A_i\}$ of sets from $\mathcal{A}$, we have $\bigcup A_i \in \mathcal{A}$.

De Morgan's laws imply that countable intersections stay in $\mathcal{A}$.

Theorem

There is a smallest σ-algebra containing any collection of sets.

Sidebar: Borel Sets

Definition

The *Borel σ-algebra on* $\mathbb{R}$ is the smallest σ-algebra containing $\mathcal{G}$, all of the open sets in $\mathbb{R}$, and is denoted by $\mathcal{B}(\mathbb{R})$.

Proposition

The Borel σ-algebra $\mathcal{B}(\mathbb{R})$ is (also) generated by each of:

- $\mathcal{F} = \{$*all closed sets in* $\mathbb{R}\}$
- $\{(-\infty, b] : b \in \mathbb{R}\}$
- $\{(a, b] : a, b \in \mathbb{R}\}$

Proposition

Let $\mathcal{S}_\delta = \{\bigcap S_i : S_i \in \mathcal{S}\}$ *and* $\mathcal{S}_\sigma = \{\bigcup S_i : S_i \in \mathcal{S}\}$. *Then*

$$\begin{array}{l} \mathcal{G} \subsetneq \mathcal{G}_\delta \subsetneq \mathcal{G}_{\delta\sigma} \subsetneq \mathcal{G}_{\delta\sigma\delta} \subsetneq \cdots \\ \quad \Updownarrow \quad \Updownarrow \quad \Updownarrow \quad \Updownarrow \qquad \cdots \subsetneq \mathcal{B}(\mathbb{R}) \subsetneq \mathcal{P}(\mathbb{R}) \\ \mathcal{F} \subsetneq \mathcal{F}_\sigma \subsetneq \mathcal{F}_{\sigma\delta} \subsetneq \mathcal{F}_{\sigma\delta\sigma} \subsetneq \cdots \end{array}$$

Countably Additive Measure

Definition
A *countably additive measure* is a set function m such that

- m is a non-negative extended real-valued function on a σ-algebra $\mathfrak{M}$ of subsets of $\mathbb{R}$; that is, $m : \mathfrak{M} \to [0, \infty]$.
- $m\left(\bigcup E_n\right) = \sum m(E_n)$ for any sequence of disjoint subsets.

Exercises
Let m be a countably additive measure on the σ-algebra $\mathfrak{M}$.

1. *If A and B are in $\mathfrak{M}$ with $A \subset B$, then $m(A) \leq m(B)$.*
2. *If there is a set $A \in \mathfrak{M}$ with $m(A) < \infty$, then $m(\emptyset) = 0$.*
3. *Show that m is countably subadditive or that for any sequence of sets, $m(\bigcup E_n) \leq \sum m(E_n)$. (Hint: $B_n = A_n - \bigcup_{i<n} A_i$.)*
4. *Let n be the counting measure, the number of elements in a set. Show that n satisfies Goals $1, 3,$ and 4.*

Outer Measure

Definition
The *outer measure* of A is

$$m^*(A) = \inf_{A \subset \bigcup I_n} \sum_n \ell(I_n)$$

where I_n is open and $\bigcup I_n$ covers A with a countable union.

Proposition
The outer measure of an interval is its length or $m^*(I) = \ell(I)$.

Proof.
I. $I = [a, b]$. (a) Since $[a, b] \subset (a - \epsilon, b + \epsilon)$, then $m^*(I) \leq b - a$.
(b) Heine-Borel thm: we need only consider finite covers. Work with the finite cover to show $\sum \ell(I_n) \geq b - a$.
II. Any finite interval I. There is a closed interval $J \subset I$ such that $\ell(I) - \epsilon \leq \ell(J) = m^*(J) \leq m^*(I) \leq m^*(\bar{I}) = \ell(I)$.
III. Any infinite interval. ✓ □

Outer Measure is Countably Subadditive

Theorem
Let $\{A_n\}$ be a countable collection of subsets of $\mathbb{R}$. Then

$$m^*\left(\bigcup_n A_n\right) \leq \sum_n m^*(A_n)$$

Proof.
"Wolog" all A_n's have finite outer measure. For each A_n there is a countable collection of open intervals $\{I_{n,i}\}$ covering A_n such that

$$\sum_i \ell(I_{n,i}) < m^*(A_n) + \frac{\epsilon}{2^n}$$

The set $\{I_{n,i} : n, i \in \mathbb{N}\}$ covers $\bigcup A_n$. Thence

$$m^*\left(\bigcup_n A_n\right) \leq \sum_{n,i} \ell(I_{n,i}) = \sum_n \sum_i \ell(I_{n,i}) < \sum_n \left(m^*(A_n) + \frac{\epsilon}{2^n}\right)$$

□

Measured Exercises

Exercises

1. *If A is a countable set, then $m^*(A) = 0$.*
2. *The closed interval $[0, 1]$ is not countable.*
3. *Show that $m^*(\mathbb{Q} \cap [0, 1]) = 0$ and $m^*(\mathbb{Q}) = 0$.*
4. *Let $A = \mathbb{Q} \cap [0, 1]$ and let $\{I_n : n = 1..N\}$ be a finite collection of open intervals covering A. Then $\sum \ell(I_n) \geq 1$.*
5. *Reconcile $1.$ through $4.$*
6. *Given any set A and any $\epsilon > 0$, there is an open set G such that $A \subset G$ and $m^*(G) \leq m^*(A) + \epsilon$. (Confer "Littlewood's Three Principles.")*
7. *Why is m^* translation invariant?*

Lebesgue Measure

Lebesgue outer measure m^* satisfies goals $1, 2,$ and 4, but not goal 3, countable additivity; m^* is only countably subadditive. We can gain countable additivity by giving up goal 1 and reducing the collection $\mathfrak{M}$ of sets; there will be sets that can't be measured. This approach is not without difficulties, though. The existence of nonmeasurable sets[2] leads to problems such as Vitali's theorem which yields a method of decomposing the interval $[0, 1]$ into a set of measure 2. (Also see the Hausdorff paradox.)

We will use the definition of a set being measurable that was given by Carathéodory.

[2] See "Non-measurable_set" for an intuitive explanation.

Measurable Sets

Definition
The set E is *measurable* iff for each set A we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c).$$

Proposition
If E is measurable, then then E^c is measurable.

Proposition
If $m^*(E) = 0$, then E is measurable.

Proof.
Let A be any set. Then $A \cap E \subset E$ implies $m^*(A \cap E) \leq m^*(E)$. Hence $m^*(A \cap E) = 0$. Now $A \cap E^c \subset A$, so

$$m^*(A) \geq m^*(A \cap E^c) = m^*(A \cap E) + m^*(A \cap E^c).$$ □

Properties of "Measurable"

Proposition
If E_1 and E_2 are measurable, then so is $E_1 \cup E_2$.

Proof.
Let A be any set. Since $E_2 \in \mathfrak{M}$, then

$$m^*(A \cap E_1^c) = m^*((A \cap E_1^c) \cap E_2) + m^*((A \cap E_1^c) \cap E_2^c).$$

From $A \cap (E_1 \cup E_2) = (A \cap E_1) \cup (A \cap E_2 \cap E_1^c)$, we see that

$$m^*(A \cap (E_1 \cup E_2)) \leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c)$$

So

$$\begin{aligned} m^*(A \cap (E_1 \cup E_2)) + m^*(A \cap (E_1 \cup E_2)^c) \\ &\leq m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*(A \cap (E_1 \cup E_2)^c) \\ &= m^*(A \cap E_1) + m^*(A \cap E_1^c) = m^*(A) \end{aligned}$$ □

Proposition
$\mathfrak{M}$ is an algebra of sets.

A Bigger "Measurable" Cup

Proposition
Let A be any set and $E_1, E_2, \ldots, E_N$ be a finite sequence of disjoint measurable sets. Then

$$m^*\left(A \cap \left[\bigcup_{i=1}^{N} E_i\right]\right) = \sum_{i=1}^{N} m^*(A \cap E_i)$$

Proof.
Induction on n with $\left(A \cap \bigcup^{n} E_i\right) \cap E_n = A \cap E_n$ and $\left(A \cap \bigcup^{n} E_i\right) \cap E_n^c = A \cap \left(\bigcup^{n-1} E_i\right)$. □

A Countable “Measurable” Cup

Proposition

Let $E_1, E_2, \ldots$ be a countable sequence of measurable sets. Then $E = \bigcup_{i=1}^{\infty} E_i$ is measurable.

Proof.

Wolog the E_i are pairwise disjoint. (Otherwise define $B_i = E_i - \bigcup_{j=1}^{i-1} E_j$.) Let A be any set and set $F_n = \bigcup_{i=1}^{n} E_i$. Then $F_n \in \mathfrak{M}$ and $F_n^c \supset E^c$. Then $m^*(A \cap F_n) = \sum_{i=1}^{n} m^*(A \cap E_i)$. Hence, since n is arbitrary and $m^*(A)$ is independent of n,

$$m^*(A) \geq \sum_{i=1}^{n} m^*(A \cap E_i) + m^*(A \cap E^c) \geq m^*(A \cap E) + m^*(A \cap E^c)$$

□

Proposition

$\mathfrak{M}$ is a σ-algebra of sets.

The Lebesgue Measure m.

Definition

Define Lebesgue measure to be the restriction $m = m^*|_{\mathfrak{M}}$.

Theorem

The Borel sets are Lebesgue measurable.

Theorem

Let E be a set and let $\epsilon > 0$. TFAE:

1. *E is Lebesgue measurable*
2. *there is an open set $E \subset G$ such that $m^*(G - E) < \epsilon$*
3. *there is a closed set $F \subset E$ such that $m^*(E - F) < \epsilon$*
4. *there is a $G \in \mathcal{G}_\delta$ such that $E \subset G$ and $m^*(G - E) = 0$*
5. *there is an $F \in \mathcal{F}_\sigma$ such that $F \subset E$ and $m^*(E - F) = 0$*

Measure Zero

Definition

A set $S \subset \mathbb{R}$ has *measure zero* if and only if $m(S) = 0$; i.e., for any $\epsilon > 0$ there is an open cover $\mathcal{C} = \{G_k \mid k \in \mathbb{N}\}$ of S such that $\sum_{k \in \mathbb{N}} m(G_k) < \epsilon$.

Example

1. Any finite set (countable set) has measure zero.
2. Every interval $[a, b]$ is not measure zero (when $a < b$).

The length of $[0, 1]$ is 1. The rationals contained in $[0, 1]$ have measure zero. What is the measure of the irrationals in $[0, 1]$?

Definition (A.E.)

A property that holds for all x except on a set of measure zero is said to hold *almost everywhere.*

Sidebar: $\mathbb{Q}$ Is Small

Theorem

The rationals are countable.

Proof.

Let $\mathbb{Q}$ be the set of rational numbers. The array below shows a method of enumerating all elements of $\mathbb{Q}$.

$$\begin{array}{ccccc} 1/1_{(1)} & 2/1_{(2)} & 3/1_{(4)} & 4/1_{(7)} & \ldots \\ 1/2_{(3)} & 2/2_{(5)} & 3/2_{(8)} & 4/2_{(12)} & \ldots \\ 1/3_{(6)} & 2/3_{(9)} & 3/3_{(13)} & 4/3_{(18)} & \ldots \\ 1/4_{(10)} & 2/4_{(14)} & 3/4_{(19)} & 4/4_{(25)} & \ldots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Since each rational is counted, we have $|\mathbb{Q}| \leq |\mathbb{N}|$ where we use $|\cdot|$ to indicate cardinality (or size). But we know that $\mathbb{N} \subseteq \mathbb{Q}$, so that $|\mathbb{N}| \leq |\mathbb{Q}|$. Hence $|\mathbb{Q}| = |\mathbb{N}|$. □

Covering $\mathbb{Q}$

Theorem

The set of rationals has measure zero.

Proof.

Let $\epsilon > 0$. *List the rationals in order* $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ *as given by the "countability matrix" defined earlier. For each rational* r_k, *define the open interval* $I_k = (r_k - \epsilon/2^{k+1}, r_k + \epsilon/2^{k+1})$. *Then*

- *the collection* $\mathcal{C} = \{I_k \mid k \in \mathbb{N}\}$ *forms an open cover of* $\mathbb{Q}$,
- *the length of each* I_k *is* $m(I_k) = \epsilon/2^k$.

Then $m(\mathbb{Q}) \le m(\mathcal{C})$ *which is*

$$m(\mathbb{Q}) \le m(\mathcal{C}) = \sum_{k=1}^{\infty} m(I_k) = \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = \epsilon$$

□

Measurable Functions

Proposition (measurability condition)

Let f *be an extended real-valued function on a measurable domain* D. *Then TFAE:*

1. *For each* $\alpha \in \mathbb{R}$, *the set* $\{x : f(x) > \alpha\}$ *is measurable.*
2. *For each* $\alpha \in \mathbb{R}$, *the set* $\{x : f(x) \ge \alpha\}$ *is measurable.*
3. *For each* $\alpha \in \mathbb{R}$, *the set* $\{x : f(x) < \alpha\}$ *is measurable.*
4. *For each* $\alpha \in \mathbb{R}$, *the set* $\{x : f(x) \le \alpha\}$ *is measurable.*

These imply

5. *For each* $\beta \in \mathbb{R}_\infty$, *the set* $\{x : f(x) = \beta\}$ *is measurable.*

Proof.

$(1.) \implies (2.)\ \{x : f(x) \ge \alpha\} = \bigcap \{x : f(x) > \alpha - 1/n\}$

$(2.) \implies (3.)\ \{x : f(x) < \alpha\} = D - \{x : f(x) \ge \alpha\}$

$(3.) \implies (4.)\ \{x : f(x) \le \alpha\} = \bigcap \{x : f(x) < \alpha + 1/n\}$

$(4.) \implies (1.)\ \{x : f(x) > \alpha\} = D - \{x : f(x) \le \alpha\}$

$(*.) \implies (5.)$ Exercise. (2 cases: $\beta < \infty$ and $\beta = \pm\infty$.) □

Definition of a Measurable Function

Definition

Let D be measurable. Then $f : D \to \mathbb{R}_\infty$ is measurable iff f satisfies the *measurability condition*.

Proposition

Let f *and* g *be measurable (real-valued) functions defined on* D *and* $c \in \mathbb{R}$. *Then* $f + c$, cf, $f \pm g$, f^2, *and* fg *are measurable.*

Proof (*sketch*).

$(f + c, cf)$: Use $\{x : f(x) + c < \alpha\} = \{x : f(x) < \alpha - c\}$, etc.

$(f + g)$: If $f(x) + g(x) < \alpha$, there is an $r \in \mathbb{Q}$ ($r = r(\alpha) \ne r(x)$) such that $f(x) < r < \alpha - g(x)$. Thus

$$\{x : f(x) + g(x) < \alpha\} = \bigcup_r (\{x : f(x) < r\} \cap \{x : g(x) < \alpha - r\})$$

is a countable union of measurable sets, hence is measurable.

(f^2): Use $\{x : f^2(x) > \alpha\} = \{x : f(x) > \sqrt{\alpha}\} \cup \{x : f(x) < -\sqrt{\alpha}\}$.

(fg): Use $fg = \frac{1}{4}(f + g)^2 - \frac{1}{4}(f - g)^2$. □

Sequences of Measurable Functions

Theorem

Let $\{f_n\}$ *be a sequence of measurable functions on a common domain* D. *Then the functions*

$$\sup\{f_1, \ldots, f_n\}, \quad \sup_n f_n, \quad \limsup_n f_n$$

are measurable. Analogous statements hold for $\inf$ *and* $\liminf$.

Proof.

Set $h = \sup\{f_1, \ldots, f_n\}$, then

$$\{x : h(x) > \alpha\} = \bigcup_{i=1}^{n} \{x : f_i(x) > \alpha\}.$$

Hence h is measurable. Now set $g = \sup_n f_n$, then

$$\{x : g(x) > \alpha\} = \bigcup_{i=1}^{\infty} \{x : f_i(x) > \alpha\}.$$

Hence g is measurable. Combine the above with the definition $\limsup_n f_n = \inf_n \sup_{k \ge n} f_k$ to finish. □

‘Simple’ Functions are Measurable

Proposition

If f is measurable and $g = f$ a.e., then g is measurable.

Proof.

Set $E = \{x : f(x) \neq g(x)\}$. Then $m(E) = 0$. So $\{x : g(x) > \alpha\}$ $= \{x : f(x) > \alpha\} \cup \{x \in E : g(x) > \alpha\} - \{x \in E : g(x) \le \alpha\}$. □

Definition

A measurable real-valued function ϕ is *simple* if it assumes only finitely many values. Then

$$\phi(x) = \sum_{k=1}^{n} \alpha_k \chi_{A_k}(x) \quad \text{where} \quad A_k = \{x : \phi(x) = \alpha_k\}$$

If each A_k is an interval, then ϕ is called a *step function.*

Example

- $s(x) = \sum_{k=1}^{N} \frac{k^2}{N^2} \chi_{[\frac{k-1}{N}, \frac{k}{N}]}(x)$ is a step function; $\chi_{\mathbb{Q}}$ is simple.

Measurable Functions are ‘Simple’

Proposition

Let $f : [a, b] \to \mathbb{R}_\infty$ be measurable such that $m(\{f(x) = \pm\infty\})$ is zero. Given $\epsilon > 0$, there is a step function s and a continuous function h so that $|f - s| < \epsilon$ and $|f - h| < \epsilon$ a.e.

Proof (*Exercise*).

1. There is an M such that $|f| \le M$ except on a set of measure $< \epsilon/3$.
2. There is a simple function ϕ such that $|f - \phi| < \epsilon$ except when $|f| > M$. (*Hint:* $(M - {}^{-}M) \le n \cdot \epsilon$.)
3. There is a step function g such that $g = \phi$ except on a set of measure $< \epsilon/3$. (*Hint: look here.*)
4. There is a continuous function h such that $h = g$ except on a set of measure $< \epsilon/3$. (*Hint: think like a spline.*)

□

Functionally Measured Exercises

Exercises

1. *Let ϕ_1 and ϕ_2 be simple functions and $c \in \mathbb{R}$. Show that*
   a. *$c\phi$ is a simple function,*
   b. *$\phi_1 + \phi_2$ is a simple function,*
   c. *$\phi_1 \cdot \phi_2$ is a simple function.*
2. *For a set S define the characteristic or indicator function to be $\chi_S(x) = \begin{cases} 1 & x \in S \\ 0 & x \notin S \end{cases}$. Show that*
   a. $\chi_{A \cap B} = \chi_A \cdot \chi_B$,
   b. $\chi_{A \cup B} = \chi_A + \chi_B - \chi_A \cdot \chi_B$.
   c. $\chi_{A^c} = 1 - \chi_A$.
3. *Let D be a dense set of real numbers; i.e., every interval contains an element of D. Let f be an extended real-valued function on $\mathbb{R}$ such that for any $d \in D$, the set $\{x : f(x) > d\}$ is measurable. Then f is measurable.*

Integration

We began by looking at two examples of integration problems.

- The Riemann integral over $[0, 1]$ of a function with infinitely many discontinuities didn’t exist even though the points of discontinuity formed a set of measure zero.
  (The points of discontinuity formed a dense set in $[0, 1]$.)
- The limit of a sequence of Riemann integrable functions did not equal the integral of the limit function of the sequence.
  (Each function had area $^1/_2$, but the limit of the sequence was the zero function.)

We will look at Riemann integration, then Riemann-Stieltjes integration, and last, develop the Lebesgue integral.

There are many other types of integrals: Darboux, Denjoy, Gauge, Perron, etc. See the list given in the “See also” section of *Integrals* on Mathworld.

Riemann Integral

Definition

- A *partition* $\mathcal{P}$ of $[a,b]$ is a finite set of points such that $\mathcal{P} = \{a = x_0 < x_1 < \cdots < x_{n-1} < x_n = b\}$.
- Set $M_i = \sup f(x)$ on $[x_{i-1}, x_i]$. The *upper sum* of f on $[a,b]$ w.r.t. $\mathcal{P}$ is

$$U(\mathcal{P}, f) = \sum_{i=1}^{n} M_i \cdot \Delta x_i$$

- The *upper Riemann integral* of f over $[a,b]$ is

$$\overline{\int_a^b} f(x)\,dx = \inf_{\mathcal{P}} U(\mathcal{P}, f)$$

Exercise

1. *Define the lower sum $L(\mathcal{P}, f)$ and the lower integral $\underline{\int_a^b} f$.*

Definitely a Riemann Integral

Definition

If $\overline{\int_a^b} f(x)\,dx = \underline{\int_a^b} f(x)\,dx$, then f is Riemann integrable and is written as $\int_a^b f(x)\,dx$ and $f \in \mathfrak{R}$ on $[a,b]$.

Proposition

A function f is Riemann integrable on $[a,b]$ if and only if for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a,b]$ such that

$$U(\mathcal{P}, f) - L(\mathcal{P}, f) < \epsilon.$$

Theorem

If f is continuous on $[a,b]$, then $f \in \mathfrak{R}$ on $[a,b]$.

Theorem

If f is bounded on $[a,b]$ with only finitely many points of discontinuity, then $f \in \mathfrak{R}$ on $[a,b]$.

Properties of Riemann Integrals

Proposition

Let f and $g \in \mathfrak{R}$ on $[a,b]$ and $c \in \mathbb{R}$. Then

- $\int_a^b cf\,dx = c\int_a^b f\,dx$
- $\int_a^b (f+g)\,dx = \int_a^b f\,dx + \int_a^b g\,dx$
- $f \cdot g \in \mathfrak{R}$
- *if $f \le g$, then $\int_a^b f\,dx \le \int_a^b g\,dx$*
- $\left|\int_a^b f\,dx\right| \le \int_a^b |f|\,dx$
- *Define $F(x) = \int_a^x f(t)\,dt$. Then F is continuous and, if f is continuous at x_0, then $F'(x_0) = f(x_0)$*
- *If $F' = f$ on $[a,b]$, then $\int_a^b f(x)\,dx = F(b) - F(a)$*

Riemann Integrated Exercises

Exercises

1. *If $\int_a^b |f(x)|\,dx = 0$, then $f = 0$.*
2. *Show why $\int_0^1 \chi_{\mathbb{Q}}(x)\,dx$ does not exist.*
3. *Define*

$$S_n(x) = \sum_{k=1}^{n+1} \left(\frac{k-1}{k} \cdot \chi_{[\frac{k-1}{k}, \frac{k}{k+1})}(x) \right) + \frac{n}{n+1} \chi_{[\frac{n+1}{n+2}, 1]}(x).$$

   3.1 *How many discontinuities does S_n have?*
   3.2 *Prove that $S_n'(x) = 0$ a.e.*
   3.3 *Calculate $\int_0^1 S_n(x)\,dx$.*
   3.4 *What is S_∞?*
   3.5 *Does $\int_0^1 S_\infty(x)\,dx$ exist?*

(See an animated graph of S_N.)

Riemann-Stieltjes Integral

Definition

- Let $\alpha(x)$ be a monotonically increasing function on $[a, b]$. Set $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.
- Set $M_i = \sup f(x)$ on $[x_{i-1}, x_i]$. The *upper sum* of f on $[a, b]$ w.r.t. α and $\mathcal{P}$ is

$$U(\mathcal{P}, f, \alpha) = \sum_{i=1}^{n} M_i \cdot \Delta\alpha_i$$

- The *upper Riemann-Stieltjes integral* of f over $[a, b]$ w.r.t. α is

$$\overline{\int_a^b} f(x)\, d\alpha(x) = \inf_{\mathcal{P}} U(\mathcal{P}, f, \alpha)$$

Exercise

1. *Define the lower sum* $L(\mathcal{P}, f, \alpha)$ *and lower integral* $\underline{\int_a^b} f\, d\alpha$.

Definitely a Riemann-Stieltjes Integral

Definition

If $\overline{\int_a^b} f\, d\alpha = \underline{\int_a^b} f\, d\alpha$, then f is Riemann-Stieltjes integrable and is written as $\int_a^b f(x)\, d\alpha(x)$ and $f \in \mathfrak{R}(\alpha)$ on $[a, b]$.

Proposition

A function f is Riemann-Stieltjes integrable w.r.t. α on $[a, b]$ iff for every $\epsilon > 0$ there is a partition $\mathcal{P}$ of $[a, b]$ such that

$$U(\mathcal{P}, f, \alpha) - L(\mathcal{P}, f, \alpha) < \epsilon.$$

Theorem

If f is continuous on $[a, b]$, then $f \in \mathfrak{R}(\alpha)$ on $[a, b]$.

Theorem

If f is bounded on $[a, b]$ with only finitely many points of discontinuity and α is continuous at each of f's discontinuities, then $f \in \mathfrak{R}(\alpha)$ on $[a, b]$.

Properties of Riemann-Stieltjes Integrals

Proposition

Let f and $g \in \mathfrak{R}(\alpha)$ and in β on $[a, b]$ and $c \in \mathbb{R}$. Then

- $\int_a^b cf\, d\alpha = c\int_a^b f\, d\alpha$ *and* $\int_a^b f\, d(c\alpha) = c\int_a^b f\, d\alpha$
- $\int_a^b (f+g)\, d\alpha = \int_a^b f\, d\alpha + \int_a^b g\, d\alpha$ *and* $\int_a^b f\, d(\alpha+\beta) = \int_a^b f\, d\alpha + \int_a^b f\, d\beta$
- $f \cdot g \in \mathfrak{R}(\alpha)$
- *if* $f \le g$, *then* $\int_a^b f\, d\alpha \le \int_a^b g\, d\alpha$
- $\left|\int_a^b f\, d\alpha\right| \le \int_a^b |f|\, d\alpha$
- *Suppose that $\alpha' \in \mathfrak{R}$ and f is bounded. Then $f \in \mathfrak{R}(\alpha)$ iff $f\alpha' \in \mathfrak{R}$ and*

$$\int_a^b f\, d\alpha = \int_a^b f \cdot \alpha'\, dx$$

Riemann-Stieltjes Integrals and Series

Proposition

If f is continuous at $c \in (a, b)$ and $\alpha(x) = r$ for $a \le x < c$ and $\alpha(x) = s$ for $c < x \le b$, then

$$\begin{aligned}\int_a^b f\, d\alpha &= f(c)\,(\alpha(c+) - \alpha(c-))\\ &= f(c)\,(s - r)\end{aligned}$$

Proposition

Let $\alpha = \lfloor x \rfloor$, the greatest integer function. If f is continuous on $[0, b]$, then

$$\int_0^b f(x)\, d\lfloor x \rfloor = \sum_{k=1}^{\lfloor b \rfloor} f(k)$$

Riemann-Stieltjes Integrated Exercises

Exercises

1. $\int_0^1 x\,dx^2$
2. $\int_0^{\pi/2} \cos(x)\,d\sin(x)$
3. $\int_0^{5/2} x\,d(x - \lfloor x \rfloor)$
4. $\int_{-1}^1 e^x\,d|x|$
5. $\int_{-3/2}^{3/2} e^x\,d\lfloor x \rfloor$
6. $\int_{-1}^1 e^x\,d\lfloor x \rfloor$
7. *Set H to be the Heaviside function; i.e.,*

$$H(x) = \begin{cases} 0 & x \le 0 \\ 1 & \textit{otherwise} \end{cases}.$$

Show that, if f is continuous at 0, then

$$\int_{-\infty}^{+\infty} f(x)\,dH(x) = f(0).$$

Lebesgue Integral

We start with simple functions.

Definition

A function has *finite support* if it vanishes outside a finite interval.

Definition

Let ϕ be a measurable simple function with finite support. If $\phi(x) = \sum_{i=1}^{n} a_i \chi_{A_i}(x)$ is a representation of ϕ, then

$$\int \phi(x)\,dx = \sum_{i=1}^{n} a_i \cdot m(A_i)$$

Definition

If E is a measurable set, then $\int_E \phi = \int \phi \cdot \chi_E$.

Integral Linearity

Proposition

If ϕ and ψ are measurable simple functions with finite support and $a, b \in \mathbb{R}$, then $\int (a\phi + b\psi) = a \int \phi + b \int \psi$. Further, if $\phi \le \psi$ a.e., then $\int \phi \le \int \psi$.

Proof (sketch).

I. Let $\phi = \sum^{N} \alpha_i \chi_{A_i}$ and $\psi = \sum^{M} \beta_i \chi_{B_i}$. Then show $a\phi + b\psi$ can be written as $a\phi + b\psi = \sum^{K} (a\alpha_{k_i} + b\beta_{k_j})\chi_{E_k}$ for the properly chosen E_k. Set A_0 and B_0 to be zero sets of ϕ and ψ. (Take $\{E_k : k = 0..K\} = \{A_j \cap B_k : j = 0..N, k = 0..M\}$.)

II. Use the definition to show $\int \psi - \int \phi = \int (\psi - \phi) \ge \int 0 = 0$. □

Steps to the Lebesgue Integral

Proposition

Let f be bounded on $E \in \mathfrak{M}$ with $m(E) < \infty$. Then f is measurable iff

$$\inf_{f \le \psi} \int_E \psi = \sup_{f \ge \phi} \int_E \phi$$

for all simple functions ϕ and ψ.

Proof.

I. Suppose f is bounded by m. Define

$$E_k = \left\{ x : \frac{k-1}{n} M < f(x) \le \frac{k}{n} M \right\}, \qquad -n \le k \le n$$

The E_k are measurable, disjoint, and have union E. Set

$$\psi_n(x) = \frac{M}{n} \sum_{-n}^{n} k\,\chi_{E_k}(x), \quad \phi_n(x) = \frac{M}{n} \sum_{-n}^{n} (k-1)\,\chi_{E_k}(x)$$

SLI (cont)

(proof cont).

Then $\phi_n(x) \leq f(x) \leq \psi(x)$, and so

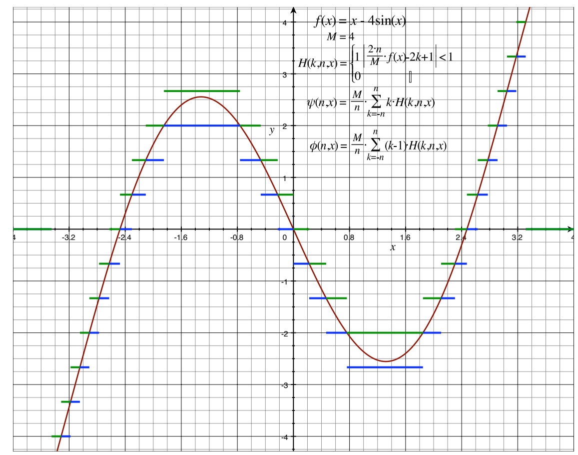
- $\inf \int_E \psi \leq \int_E \psi_n = \frac{M}{n} \sum_{k=-n}^{n} k\, m(E_k)$
- $\sup \int_E \phi \geq \int_E \phi_n = \frac{M}{n} \sum_{k=-n}^{n} (k-1)\, m(E_k)$

Thus $0 \leq \inf \int_E \psi - \sup \int_E \phi \leq \frac{M}{n} m(E)$. Since n is arbitrary, equality holds.

II. Suppose that $\inf \int_E \psi = \sup \int_E \phi$. Choose ϕ_n and ψ_n so that $\phi_n \leq f \leq \psi_n$ and $\int_E (\psi_n - \phi_n) < \frac{1}{n}$. The functions $\psi^* = \inf \psi_n$ and $\phi^* = \sup \phi_n$ are measurable and $\phi^* \leq f \leq \psi^*$. The set $\Delta = \{x : \phi^*(x) < \psi^*(x)\}$ has measure 0. Thus $\phi^* = \psi^*$ almost everywhere, so $\phi^* = f$ a.e. Hence f is measurable. □

Example Steps

Example



Defining the Lebesgue Integral

Definition

If f is a bounded measurable function on a measurable set E with $m(E) < \infty$, then

$$\int_E f = \inf_{\psi \geq f} \int_E \psi$$

for all simple functions $\psi \geq f$.

Proposition

Let f be a bounded function defined on $E = [a, b]$. If f is Riemann integrable on $[a, b]$, then f is measurable on $[a, b]$ and

$$\int_E f = \int_a^b f(x)\, dx;$$

the Riemann integral of f equals the Lebesgue integral of f.

Properties of the Lebesgue Integral

Proposition

If f and g are measurable on E, a set of finite measure, then

- $\int_E (\alpha f + \beta g) = \alpha \int_E f + \beta \int_E g$
- *if $f = g$ a.e., then* $\int_E f = \int_E g$
- *if $f \leq g$ a.e., then* $\int_E f \leq \int_E g$
- $\left| \int_E f \right| \leq \int_E |f|$
- *if $a \leq f \leq b$, then* $a \cdot m(E) \leq \int_E f \leq b \cdot m(E)$
- *if $A \cap B = \emptyset$, then* $\int_{A \cup B} f = \int_A f + \int_B f$

Proof.

Exercise. □

Lebesgue Integral Examples

Examples

1. Let $D(x) = \begin{Bmatrix} \frac{1}{q} & x = \frac{p}{q} \in \mathbb{Q} \\ 0 & \text{otherwise} \end{Bmatrix}$. Then $\int_{[0,1]} D = \int_0^1 D(x)\,dx$.
2. Let $\chi_{\mathbb{Q}}(x) = \begin{Bmatrix} 1 & x \in \mathbb{Q} \\ 0 & \text{otherwise} \end{Bmatrix}$. Then $\int_{[0,1]} \chi_{\mathbb{Q}} \neq \int_0^1 \chi_{\mathbb{Q}}(x)dx$.
3. Define
$$f_n(x) = \sum_{k=1}^{n+1} \left(\frac{k-1}{k} \cdot \chi_{[\frac{k-1}{k}, \frac{k}{k+1})}(x) \right) + \frac{n}{n+1} \chi_{[\frac{n+1}{n+2}, 1]}(x).$$
Then
   3.1 f_n is a step function, hence integrable
   3.2 $f_n'(x) = 0$ a.e.
   3.3 $\frac{1}{4} \leq \int_{[0,1]} f_n = \int_0^1 f_n(x)\,dx < \frac{3}{8}$

Extending the Integral Definition

Definition

Let f be a nonnegative measurable function defined on a measurable set E. Define
$$\int_E f = \sup_{h \leq f} \int_E h$$
where h is a bounded measurable function with finite support.

Proposition

If f and g are nonnegative measurable functions, then

- $\int_E c\,f = c \int_E f$ for $c > 0$
- $\int_E f + g = \int_E f + \int_E g$
- *If $f \leq g$ a.e., then* $\int_E f \leq \int_E g$

Proof.

Exercise. □

General Lebesgue's Integral

Definition

Set $f^+(x) = \max\{f(x), 0\}$ and $f^-(x) = \max\{-f(x), 0\}$. Then $f = f^+ - f^-$ and $|f| = f^+ + f^-$. A measurable function f is integrable over E iff both f^+ and f^- are integrable over E, and then $\int_E f = \int_E f^+ - \int_E f^-$.

Proposition

Let f and g be integrable over E and let $c \in \mathbb{R}$. Then

1. $\int_E cf = c \int_E f$
2. $\int_E f + g = \int_E f + \int_E g$
3. *if $f \leq g$ a.e., then* $\int_E f \leq \int_E g$
4. *if A, B are disjoint m'ble subsets of E,* $\int_{A \cup B} f = \int_A f + \int_B f$

Convergence Theorems

Theorem (Bounded Convergence Theorem)

Let $\{f_n : E \to \mathbb{R}\}$ be a sequence of measurable functions converging to f with $m(E) < \infty$. If there is a uniform bound M for all f_n, then
$$\int_E \lim_n f_n = \lim_n \int_E f_n$$

Proof (sketch).

Let $\epsilon > 0$.

1. f_n converges "almost uniformly;" i.e., $\exists A, N$ s.t. $m(A) < \frac{\epsilon}{4M}$ and, for $n > N, x \in E - A \implies |f_n(x) - f(x)| \leq \frac{\epsilon}{2\,m(E)}$.
2. $\left| \int_E f_n - \int_E f \right| = \left| \int_E f_n - f \right| \leq \int_E |f_n - f| = \left(\int_{E-A} + \int_A \right) |f_n - f|$
3. $\int_{E-A} |f_n - f| + \int_A |f_n| + |f| \leq \frac{\epsilon}{2m(E)} \cdot m(E) + 2M \cdot \frac{\epsilon}{4M} = \epsilon$ □

Lebesgue's Dominated Convergence Theorem

Theorem (Dominated Convergence Theorem)

Let $\{f_n : E \to \mathbb{R}\}$ be a sequence of measurable functions converging a.e. on E with $m(E) < \infty$. If there is an integrable function g on E such that $|f_n| \le g$ then

$$\int_E \lim_n f_n = \lim_n \int_E f_n$$

Lemma

Under the conditions of the DCT, set $g_n = \sup_{k \ge n} \{f_n, f_{n+1}, \dots\}$ and $h_n = \inf_{k \ge n} \{f_n, f_{n+1}, \dots\}$. Then g_n and h_n are integrable and $\lim g_n = f = \lim h_n$ a.e.

Proof of DCT (sketch).

- Both g_n and h_n are monotone and converging. Apply MCT.
- $h_n \le f_n \le g_n \implies \int_E h_n \le \int_E f_n \le \int_E g_n$. □

Increasing the Convergence

Theorem (Fatou's Lemma)

If $\{f_n\}$ is a sequence of measurable functions converging to f a.e. on E, then

$$\int_E \lim_n f_n \le \liminf_n \int_E f_n$$

Theorem (Monotone Convergence Theorem)

If $\{f_n\}$ is an increasing sequence of nonnegative measurable functions converging to f, then

$$\int \lim_n f_n = \lim_n \int f_n$$

Corollary (Beppo Levi Theorem (cf.))

If $\{f_n\}$ is a sequence of nonnegative measurable functions, then

$$\int \sum_{n=1}^{\infty} f_n = \sum_{n=1}^{\infty} \int f_n$$

Sidebar: Littlewood's Three Principles

John Edensor Littlewood said,

> *The extent of knowledge required is nothing so great as sometimes supposed. There are three principles, roughly expressible in the following terms:*
>
> - *every measurable set is nearly a finite union of intervals;*
> - *every measurable function is nearly continuous;*
> - *every convergent sequence of measurable functions is nearly uniformly convergent.*
>
> *Most of the results of analysis are fairly intuitive applications of these ideas.*

From *Lectures on the Theory of Functions,* Oxford, 1944, p. 26.

Extensions of Convergence

The sequence f_n converges to f ...

Definition (Convergence Almost Everywhere)

almost everywhere if $m(\{x : f_n(x) \nrightarrow f(x)\}) = 0.$

Definition (Convergence Almost Uniformly)

almost uniformly on E if, for any $\epsilon > 0$, there is a set $A \subset E$ with $m(A) < \epsilon$ so that f_n converges uniformly on $E - A$.

Definition (Convergence in Measure)

in measure if, for any $\epsilon > 0$, $\lim_{n\to\infty} m(\{x : |f_n(x) - f(x)| \ge \epsilon\}) = 0.$

Definition (Convergence in Mean (of order $p > 1$))

in mean if $\lim_{n\to\infty} \|f_n - f\|_p = \lim_{n\to\infty} \left[\int_E |f - f_n|^p\right]^{1/p} = 0$

Integrated Exercises

Exercises

1. *Prove: If f is integrable on E, then $|f|$ is integrable on E.*
2. *Prove: If f is integrable over E, then* $\left|\int_E f\right| \leq \int_E |f|$.
3. *True or False: If $|f|$ is integrable over E, then f is integrable over E.*
4. *Let f be integrable over E. For any $\epsilon > 0$, there is a simple (resp. step) function ϕ (resp. ψ) such that* $\int_E |f - \phi| < \epsilon$.
5. *For $n = k + 2^\nu, 0 \leq k < 2^\nu$, define $f_n = \chi_{[k2^{-\nu},(k+1)2^{-\nu}]}$.*
   5.1 *Show that f_n does not converge for any $x \in [0, 1]$.*
   5.2 *Show that f_n does not converge a.e. on $[0, 1]$.*
   5.3 *Show that f_n does not converge almost uniformly on $[0, 1]$.*
   5.4 *Show that $f_n \to 0$ in measure.*
   5.5 *Show that $f_n \to 0$ in mean (of order 2).*

References

Texts on analysis, integration, and measure:

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Comparison of different types of integrals:

- *Integral, Measure, and Derivative: A Unified Approach,* G. Shilov and B. Gurevich