1 Theoretical Calculus

1.1 Limits

Definition 1 (Accumulation Point). *Let $D \subseteq \mathbb{R}$. A point $a \in \mathbb{R}$ is an* accumulation point of D *iff every open interval containing a also contains a point $x \in D$ with $x \neq a$.*

Definition 2. *Let $f : D \to \mathbb{R}$ and a be an accumulation point of D. Then*

$$\lim_{x \to a} f(x) = L$$

iff for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in D$ and $0 < |x - a| < \delta$, then $|f(x) - L| < \epsilon$.

Theorem 1 (Algebra of Limits). *Suppose that $f, g : D \to \mathbb{R}$ both have finite limits at $x = a \in D$ and $c \in \mathbb{R}$. Then*

- $\lim_{x \to a} c\, f(x) = c \lim_{x \to a} f(x)$
- $\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
- *if* $\lim_{x \to a} g(x) \neq 0$, *then* $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

Theorem 2 ("Sandwich" Theorem). *Suppose that $g(x) \leq f(x) \leq h(x)$ for all $x \in (a - h, a + h)$ for some $h > 0$. If $\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x)$, then $\lim_{x \to a} f(x) = L$.*

1.2 Continuity

Definition 3. *Let $f : D \to \mathbb{R}$ and a be an accumulation point of D. Then f* is continuous at $x = a$ *iff for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x \in D$ and $|x - a| < \delta$, then $|f(x) - f(a)| < \epsilon$.*

Theorem 3. *Every real polynomial is continuous at every $x \in \mathbb{R}$.*

Theorem 4 (Algebra of Continuity). *Suppose that $f, g : D \to \mathbb{R}$ both are continuous at $x = a \in D$ and that $c \in \mathbb{R}$. Then*

- *cf is continuous at a*
- *$f \pm g$ is continuous at a*
- *$f \cdot g$ is continuous at a*
- *if $g(a) \neq 0$, then f/g is continuous at a*

Theorem 5 (Continuity of Composition). *Let $f : A \to \mathbb{R}$ and $g : B \to \mathbb{R}$. Suppose that f is continuous at $x = a \in A$, that g is continuous at $x = f(a) \in B$, and that $f(A) \subseteq B$. Then $g \circ f$ is continuous at $x = a$.*

Theorem 6. *If a function f is continuous at a and ϕ is a function such that $\lim_{t \to t_0} \phi(t) = a$, then*

$$\lim_{t \to t_0} f\left(\phi(t)\right) = f\left(\lim_{t \to t_0} \phi(t)\right)$$

Theorem 7. *If a function f is continuous on a closed, finite interval $[a, b]$, then f is bounded on $[a, b]$.*

Theorem 8 (Intermediate Value Theorem). *If $f : [a, b] \to \mathbb{R}$ is continuous and if k is between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = k$.*

Corollary 9. *Every odd degree real polynomial has a real root.*

Theorem 10 (Extreme Value Theorem). *If $f : [a, b] \to \mathbb{R}$ is continuous, then*

1. *there exists $x_m \in [a, b]$ such that $f(x_m) = \min_{x \in [a,b]} f(x)$*
2. *there exists $x_M \in [a, b]$ such that $f(x_M) = \max_{x \in [a,b]} f(x)$*

Definition 4 (Uniform Continuity). *A function $f : D \to \mathbb{R}$ is* uniformly continuous *on D iff for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x_1, x_2 \in D$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$.*

Theorem 11. *If f is continuous on $[a, b]$, then f is uniformly continuous on $[a, b]$.*

1.3 The Derivative

Definition 5. *Let $f : D \to \mathbb{R}$ and $a \in D$ be an accumulation point. Then*

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{h \to 0} \frac{f(a+h) - f(a)}{h} \end{aligned}$$

Theorem 12. *If f is differentiable at $x = a$, then f is continuous at $x = a$.*

Theorem 13 (Algebra of Derivatives)**.** *If $f, g : D \to \mathbb{R}$ are differentiable at $x = a$ and $c \in \mathbb{R}$, then at $x = a$,*

- $(cf)' = c(f')$
- $(f \pm g)' = f' \pm g'$
- $(f \cdot g)' = f' \cdot g + f \cdot g'$
- *if* $g'(a) \neq 0$, *then* $\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g + f \cdot g'}{g^2}$

Theorem 14 (The Chain Rule)**.** *Let $f : A \to B$ and $g : B \to \mathbb{R}$. Suppose that f is differentiable at $x = a \in A$ and that g is differentiable at $x = b = f(a) \in B$. Then $g \circ f$ is differentiable at $x = a$ and*

$$(g \circ f)'(a) = g'\left(f(a)\right) \cdot f'(a)$$

Corollary 15. *Let u be a differentiable function of x and $r \in \mathbb{R}$. Then, when defined,*

$(u^r)' = r\, u^{r-1} \cdot u'$	
$(e^u)' = e^u \cdot u'$	
$\ln(u)' = \dfrac{1}{u} \cdot u'$	
$\sin(u)' = \cos(u) \cdot u'$	$\cos(u)' = -\sin(u) \cdot u'$
$\tan(u)' = \sec^2(u) \cdot u'$	$\cot(u)' = -\csc^2(u) \cdot u'$
$\sec(u)' = \sec(u)\tan(u) \cdot u'$	$\csc(u)' = -\csc(u)\cot(u) \cdot u'$
$\sin^{-1}(u)' = \dfrac{1}{\sqrt{1-u^2}} \cdot u'$	$\cos^{-1}(u)' = \dfrac{-1}{\sqrt{1-u^2}} \cdot u'$
$\tan^{-1}(u)' = \dfrac{1}{1+u^2} \cdot u'$	$\cot^{-1}(u)' = \dfrac{-1}{1+u^2} \cdot u'$
$\sec^{-1}(u)' = \dfrac{1}{\lvert u\rvert\sqrt{u^2-1}} \cdot u'$	$\csc^{-1}(u)' = \dfrac{-1}{\lvert u\rvert\sqrt{u^2-1}} \cdot u'$

Theorem 16 (Inverse Function Theorem)**.** *Let $f : [a, b] \to \mathbb{R}$ be differentiable with $f'(x) \neq 0$ for any $x \in [a, b]$. Then*

- *f is injective (*1–1*)*
- *f^{-1} is continuous on $f([a, b])$*
- *f^{-1} is differentiable on $f([a, b])$*
- $(f^{-1})'(y) = \dfrac{1}{f'(x)}$ *where $y = f(x)$*

Theorem 17. *Suppose that $f : [a, b] \to \mathbb{R}$ has an extremum at $c \in (a, b)$. If f is differentiable at $c \in (a, b)$, then $f'(c) = 0$.*

Theorem 18 (Rolle's Theorem)**.** *If $f : D \to \mathbb{R}$ is continuous on $[a, b] \subseteq D$ and differentiable on (a, b) with $f(a) = f(b)$, then there exists a value $c \in (a, b)$ such that $f'(c) = 0$.*

Theorem 19 (Mean Value Theorem)**.** *If $f : D \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a value $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

Corollary 20. *If $f : D \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on (a, b), then there exists a value $\theta \in (0, 1)$ such that*

$$f(a + h) = f(a) + h \cdot f'(a + \theta h).$$

Corollary 21. *If $f : D \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on (a, b), and $f'(x) = 0$, then f is a constant function.*

Corollary 22. *If $f, g : D \to \mathbb{R}$ are continuous on $[a, b]$, differentiable on (a, b), and $f'(x) = g'(x)$ on D, then $f(x) = g(x) + k$ on D where k is a constant.*

Corollary 23. *If $f : D \to \mathbb{R}$ is differentiable on $[a, b]$, then f' has the* Intermediate Value Property.

Theorem 24 (Cauchy's Mean Value Theorem)**.** *If $f, g : D \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on (a, b), then there exists a value $c \in (a, b)$ such that*

$$f'(c)\,[g(b) - g(a)] = g'(c)\,[f(b) - f(a)]$$

or, when denominators are non-zero,

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

Definition 6 (Uniform Differentiability)**.** *Let $f : [a, b] \to \mathbb{R}$. Then f is* uniformly differentiable on $[a, b]$ *iff f is differentiable on $[a, b]$ and, for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x_1, x_2 \in [a, b]$ with $|x_1 - x_2| < \delta$, it must follow that*

$$\left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} - f'(x_1) \right| < \epsilon$$

Corollary 25. *If $f : D \to \mathbb{R}$ is uniformly differentiable on $[a, b]$, then f' is continuous on $[a, b]$.*

Definition 7 (Lipschitz Condition)**.** *Let $f : D \to \mathbb{R}$. If there are positive constants M and α such that for any $x_1, x_2 \in D$*

$$|f(x_1) - f(x_2)| \leq M \cdot |x_1 - x_2|^{\alpha}$$

then f is Lipschitz-α *with constant M, written $f \in \operatorname{Lip}_M \alpha$.*

Theorem 26. *If $f \in \operatorname{Lip}_M \alpha$ on D, then*

1. *f is continuous,*
2. *if $\alpha > 1$, f is constant,*

Corollary 27. *If $f : [a, b] \to \mathbb{R}$ is differentiable, then $f \in \operatorname{Lip}_M 1$.*

Theorem 28 (Rademacher's Theorem)**.** *If $f \in \operatorname{Lip}_M 1$, then f is differentiable* almost everywhere.

Definition 8 (Higher Order Derivatives)**.** *The nth derivative of $f(x)$, if it exists, is given by $f^{(n)}(x) = \frac{d}{dx} f^{(n-1)}(x)$ for $n > 1$ where $f^{(0)} = f$.*

Theorem 29. *Let $f : D \to \mathbb{R}$ be m times continuously differentiable. Then f has* a root of multiplicity m at $x = r$ *iff $f^{(m)}(r) \neq 0$, but*

$$f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0$$

Theorem 30 (Taylor's Theorem or Extended Law of the Mean)**.** *Let $n \in \mathbb{N}$ and suppose that f has $n + 1$ derivatives on $(a - h, a + h)$ for some $h > 0$. Then for $x \in (a - h, a + h)$*

$$f(x) = \sum_{k=0}^{n} \frac{f^{(k)}(a)}{k!}\,(x - a)^k + R_n(x)$$

where

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n + 1)!}\,(x - a)^{n+1}$$

for some c between x and a.

Theorem 31 (L'Hôpital's Rule)**.** *Suppose that f and g are differentiable on an open interval I containing a, and that*

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$$

while $g'(x) \neq 0$ on I. Then, if the limit exists,

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

Corollary 32. *Let $n \in \mathbb{N}$. Then*

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0 \quad \textit{and} \quad \lim_{x\to\infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$$

Corollary 33. *If f is twice differentiable on an open interval I and $x \in I$, then*

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

1.4 Riemann Integration