# Types of Convergence

Let $[a, b] \subset \mathcal{R}$ be such that $m([a, b]) < \infty$. Let $\{f_n\}$ be a sequence of real-valued functions that is finite almost everywhere on $[a, b]$.

## Pointwise

The sequence $\{f_n\}$ *converges pointwise* to $f$ on $[a, b]$ if, for every $x \in [a, b]$, $\lim_n f_n(x) = f(x)$; i.e, for every $\epsilon > 0$ and $x \in [a, b]$, there is an $n^* = n^*(\epsilon, x) > 0$, such that if $n > n^*$, then $|f_n(x) - f(x)| < \epsilon$.

## Uniform

The sequence $\{f_n\}$ *converges uniformly* to $f$ on $[a, b]$ if, for every $\epsilon > 0$, there is an $n^* = n^*(\epsilon) > 0$, such that for any $x \in [a, b]$ and $n > n^*$, then $|f_n(x) - f(x)| < \epsilon$.

## Almost Everywhere

The sequence $\{f_n\}$ *converges almost everywhere* to $f$ on $[a, b]$ if $f_n$ converges pointwise to $f$ on $[a, b] \setminus E$ and $m(E) = 0$.

## Almost Uniform

The sequence $\{f_n\}$ *converges almost uniformly* to $f$ on $[a, b]$ if, for any $\epsilon > 0$ there is a set $E \subseteq [a, b]$ with $m(E) < \epsilon$ and $f_n$ converges uniformly to $f$ on $[a, b] \setminus E$.

## In Measure

The sequence $\{f_n\}$ *converges in measure* to $f$ on $[a, b]$ if, for any $\epsilon > 0$,

$$\lim_n m(\{x \in [a, b] : |f_n(x) - f(x)| > \epsilon\}) = 0$$

## In Mean

The sequence $\{f_n\}$ *converges in mean* to $f$ on $[a, b]$ if

$$\lim_n \int_{[a,b]} |f_n(x) - f(x)| \, dx = 0$$

# Problems

Find the limit $f$ and determine what type of convergence is possible.

1. Let $f_n(x) = x^n$ on $[0, 1]$ for $n = 0, 1, 2, \ldots$.
2. Let $f_1 = \chi_{[0,1]}$, $f_2 = \chi_{[0,1/2)}$, $f_3 = \chi_{[1/2,1]}$, $f_4 = \chi_{[0,1/4)}$, $f_5 = \chi_{[1/4,1/2)}$, $f_6 = \chi_{[1/2,3/4)}$, $f_7 = \chi_{[3/4,1]}$, and so forth.
3. Let $f_n$ be even, $f_n(x) = 0$ for $x > \frac{1}{n}$, and, for $0 \le x \le \frac{1}{n}$, define $f_n(x) = n \cdot (1 - nx)$. *(Who was Paul Dirac?)*

4. Let $f_n(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} - \frac{1}{2^n} \\ 1 + 2^n\left(x - \frac{1}{2}\right) & \frac{1}{2} - \frac{1}{2^n} \leq x < \frac{1}{2} \\ 1 - 2^n\left(x - \frac{1}{2}\right) & \frac{1}{2} \leq x < \frac{1}{2} + \frac{1}{2^n} \\ 0 & \frac{1}{2} + \frac{1}{2}^n \leq x \leq 1 \end{cases}$

5. Let $f_n(x) = \begin{cases} 0 & 0 \leq x < \frac{1}{2} - \frac{1}{2^n} \\ 2^{n+1} + 4^n(2x-1) & \frac{1}{2} - \frac{1}{2^n} \leq x < \frac{1}{2} \\ 2^{n+1} - 4^n(2x+1) & \frac{1}{2} \leq x < \frac{1}{2} + \frac{1}{2^n} \\ 0 & \frac{1}{2} + \frac{1}{2}^n \leq x \leq 1 \end{cases}$

Create examples or counterexamples for each of the following statements. Assume $m([a,b]) < \infty$ and $f_n$ is real-valued and finite almost everywhere.

5. $f_n$ converges almost everwhere if and only if $f_n$ converges in measure.

6. $f_n$ converges almost everwhere if and only if $f_n$ converges in mean.

7. $f_n$ converges almost uniformly if and only if $f_n$ converges in mean.

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**A Nonsequiter**

PROPOSITION. A monotone function can only have jump discontinuities.

PROOF. *WOLOG* Assume $f$ is increasing. Let $x^*$ be an interior point of $\mathrm{dom}(f)$. Let $\{x_n\}$ be a strictly monotone increasing sequence converging to $x^*$. *(Justify each of the following steps.)*

1. Then $\{f(x_n)\}$ is monotone increasing and bounded.

2. Hence $\{f(x_n)\}$ has a limit, $L_-$.

3. Suppose $x_n < x < x^*$. Then there exist an $m > n$ such that $x < x_m < x^*$. Whence, we have $f(x_n) \leq f(x) \leq f(x_m) \leq L_-$.

4. Thus $\lim_{x \nearrow x^*} f(x) = L_-$.

5. Analogously, $\lim_{x \searrow x^*} f(x) = L_+$.

6. If $L_- = L_+$, then $x^*$ is a point of continuity of $f$, otherwise $L_- < L_+$, and $f$ has a jump of $L_+ - L_-$ at $x^*$.