# Appendix A

# A Compendium of Definitions and Theorems of Elementary Real Analysis

The main definitions and results of elementary real analysis are collected here for reference. All of the theorems forming the basis of first-year calculus appear along with a number of interesting propositions that provide more depth and insight.

## A.1 LIMITS

**Definition 1 (Accumulation Point)** *Let $D \subseteq \mathbb{R}$. A point $a \in \mathbb{R}$ is an* accumulation point of $D$ *iff every open interval containing $a$ also contains a point $x \in D$ with $x \neq a$.*

**Definition 2** *Let $f : D \to \mathbb{R}$ and $a$ be an accumulation point of $D$. Then*

$$\lim_{x \to a} f(x) = L$$

*

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*iff for every* $\epsilon > 0$ *there is a* $\delta > 0$ *such that whenever* $x \in D$ *and* $0 < |x - a| < \delta$, *then* $|f(x) - L| < \epsilon$.

**Theorem 1 (Algebra of Limits)** *Suppose that* $f, g : D \to \mathbb{R}$ *both have finite limits at* $x = a \in D$ *and* $c \in \mathbb{R}$. *Then*

- $\lim_{x \to a} c\, f(x) = c \lim_{x \to a} f(x)$
- $\lim_{x \to a} f(x) \pm g(x) = \lim_{x \to a} f(x) \pm \lim_{x \to a} g(x)$
- $\lim_{x \to a} f(x) \cdot g(x) = \lim_{x \to a} f(x) \cdot \lim_{x \to a} g(x)$
- *if* $\lim_{x \to a} g(x) \neq 0$, *then* $\lim_{x \to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$

**Theorem 2 ("Sandwich Theorem")** *Suppose that* $g(x) \le f(x) \le h(x)$ *for all* $x \in (a-h, a+h)$ *for some* $h > 0$. *If* $\lim_{x \to a} g(x) = L = \lim_{x \to a} h(x)$, *then* $\lim_{x \to a} f(x) = L$.

## A.2 CONTINUITY

**Definition 3** *Let* $f : D \to \mathbb{R}$ *and* $a$ *be an accumulation point of* $D$. *Then* $f$ is continuous at $x = a$ *iff for every* $\epsilon > 0$ *there is a* $\delta > 0$ *such that whenever* $x \in D$ *and* $|x - a| < \delta$, *then* $|f(x) - f(a)| < \epsilon$.

**Theorem 3** *Every real polynomial is continuous at every* $x \in \mathbb{R}$.

**Theorem 4 (Algebra of Continuity)** *Suppose that* $f, g : D \to \mathbb{R}$ *both are continuous at* $x = a \in D$ *and that* $c \in \mathbb{R}$. *Then*

- $cf$ *is continuous at* $a$
- $f \pm g$ *is continuous at* $a$
- $f \cdot g$ *is continuous at* $a$
- *if* $g(a) \neq 0$, *then* $f/g$ *is continuous at* $a$

**Theorem 5 (Continuity of Composition)** *Let* $f : A \to \mathbb{R}$ *and* $g : B \to \mathbb{R}$ *where* $f(A) \subseteq B$. *Suppose that* $f$ *is continuous at* $x = a \in A$, *that* $g$ *is continuous at* $x = f(a) \in B$. *Then* $g \circ f$ *is continuous at* $x = a$.

**Theorem 6** *If a function* $f$ *is continuous at* $a$ *and* $\phi$ *is a function such that* $\lim_{t \to t_0} \phi(t) = a$, *then*

$$\lim_{t \to t_0} f(\phi(t)) = f\left(\lim_{t \to t_0} \phi(t)\right)$$

**Theorem 7** *If a function $f$ is continuous on a closed, finite interval $[a, b]$, then $f$ is bounded on $[a, b]$.*

**Theorem 8 (Intermediate Value Theorem)** *If $f : [a, b] \to \mathbb{R}$ is continuous and if $k$ is between $f(a)$ and $f(b)$, then there exists $c \in (a, b)$ such that $f(c) = k$.*

**Corollary 9** *Every odd degree real polynomial has a real root.*

**Corollary 10** *Every real polynomial is a product of linear factors times irreducible (over $\mathbb{R}$) quadratic factors.*

**Corollary 11 (Fundamental Theorem of Algebra)** *Every $n$th degree real polynomial has $n$ complex roots counting multiplicity.*

**Theorem 12 (Extreme Value Theorem)** *If $f : [a, b] \to \mathbb{R}$ is continuous, then*

1. *there exists $x_m \in [a, b]$ such that $f(x_m) = \min_{x \in [a,b]} f(x)$*
2. *there exists $x_M \in [a, b]$ such that $f(x_M) = \max_{x \in [a,b]} f(x)$*

**Definition 4 (Uniform Continuity)** *A function $f : D \to \mathbb{R}$ is* uniformly continuous *on $D$ iff for every $\epsilon > 0$ there is a $\delta > 0$ such that whenever $x_1, x_2 \in D$ and $|x_1 - x_2| < \delta$, then $|f(x_1) - f(x_2)| < \epsilon$.*

**Theorem 13** *If $f$ is continuous on $[a, b]$, then $f$ is uniformly continuous on $[a, b]$.*

## A.3 THE DERIVATIVE

**Definition 5** *Let $f : D \to \mathbb{R}$ and $a \in D$ be an accumulation point. Then*

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \end{aligned}$$

**Theorem 14** *If $f$ is differentiable at $x = a$, then $f$ is continuous at $x = a$.*

**Theorem 15 (Algebra of Derivatives)** *If $f, g : D \to \mathbb{R}$ are differentiable at $x = a$ and $c \in \mathbb{R}$, then at $x = a$,*

- $(cf)' = c(f')$
- $(f \pm g)' = f' \pm g'$
- $(f \cdot g)' = f' \cdot g + f \cdot g'$
- *if $g(a) \neq 0$, then* $\left(\dfrac{f}{g}\right)' = \dfrac{f' \cdot g + f \cdot g'}{g^2}$

**Theorem 16 (The Chain Rule)** *Let $f : A \to B$ and $g : B \to \mathbb{R}$. Suppose that $f$ is differentiable at $x = a \in A$ and that $g$ is differentiable at $x = b = f(a) \in B$. Then $g \circ f$ is differentiable at $x = a$ and*

$$(g \circ f)'(a) = g'\left(f(a)\right) \cdot f'(a)$$

**Corollary 17** *Let $u$ be a differentiable function of $x$ and $r \in \mathbb{R}$. Then, when defined,*

$$
\begin{array}{ll}
(u^r)' = r\, u^{r-1} \cdot u' & \\
(e^u)' = e^u \cdot u' & \ln(u)' = \dfrac{1}{u} \cdot u' \\
\hline
\sin(u)' = \cos(u) \cdot u' & \cos(u)' = -\sin(u) \cdot u' \\
\tan(u)' = \sec^2(u) \cdot u' & \cot(u)' = -\csc^2(u) \cdot u' \\
\sec(u)' = \sec(u)\tan(u) \cdot u' & \csc(u)' = -\csc(u)\cot(u) \cdot u' \\
\hline
\sin^{-1}(u)' = \dfrac{1}{\sqrt{1-u^2}} \cdot u' & \cos^{-1}(u)' = \dfrac{-1}{\sqrt{1-u^2}} \cdot u' \\
\tan^{-1}(u)' = \dfrac{1}{1+u^2} \cdot u' & \cot^{-1}(u)' = \dfrac{-1}{1+u^2} \cdot u' \\
\sec^{-1}(u)' = \dfrac{1}{|u|\sqrt{u^2-1}} \cdot u' & \csc^{-1}(u)' = \dfrac{-1}{|u|\sqrt{u^2-1}} \cdot u'
\end{array}
$$

**Theorem 18 (Inverse Function Theorem)** *Let $f : [a, b] \to \mathbb{R}$ be differentiable with $f'(x) \neq 0$ for any $x \in [a, b]$. Then*

- *$f$ is injective (1–1)*
- *$f^{-1}$ is continuous on $f([a, b])$*
- *$f^{-1}$ is differentiable on $f([a, b])$*
- $(f^{-1})'(y) = \dfrac{1}{f'(x)}$ *where $y = f(x)$*

**Theorem 19** *Suppose that $f : [a, b] \to \mathbb{R}$ has an extremum at $c \in (a, b)$. If $f$ is differentiable at $c \in (a, b)$, then $f'(c) = 0$.*

**Theorem 20 (Rolle's Theorem)** *If $f : D \to \mathbb{R}$ is continuous on $[a, b] \subseteq D$ and differentiable on $(a, b)$ with $f(a) = f(b)$, then there exists a value $c \in (a, b)$ such that $f'(c) = 0$.*

**Theorem 21 (Mean Value Theorem)** *If $f : D \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists a value $c \in (a, b)$ such that*

$$\frac{f(b) - f(a)}{b - a} = f'(c).$$

**Corollary 22** *If $f : D \to \mathbb{R}$ is continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists a value $\theta \in (0, 1)$ such that*

$$f(a + h) = f(a) + h \cdot f'(a + \theta h).$$

**Corollary 23 (Racetrack Principle)** *If $f : D \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $m \le f'(x) \le M$, then*

$$f(a) + m \cdot (b - a) \le f(b) \le f(a) + M \cdot (b - a).$$

**Corollary 24** *If $f : D \to \mathbb{R}$ is continuous on $[a, b]$, differentiable on $(a, b)$, and $f'(x) = 0$ on $D$, then $f$ is a constant function.*

**Corollary 25** *If $f, g : D \to \mathbb{R}$ are continuous on $[a, b]$, differentiable on $(a, b)$, and $f'(x) = g'(x)$ on $D$, then $f(x) = g(x) + k$ on $D$ where $k$ is a constant.*

**Corollary 26** *If $f : D \to \mathbb{R}$ is differentiable on $[a, b]$, then $f'$ has the* Intermediate Value Property.

**Theorem 27 (Cauchy's Mean Value Theorem)** *If $f, g : D \to \mathbb{R}$ are continuous on $[a, b]$ and differentiable on $(a, b)$, then there exists a value $c \in (a, b)$ such that*

$$f'(c)\,[g(b) - g(a)] = g'(c)\,[f(b) - f(a)]$$

*or, when denominators are non-zero,*

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

**Definition 6 (Uniform Differentiability)** *Let $f : [a, b] \to \mathbb{R}$. Then $f$ is* uniformly differentiable on $[a, b]$ *iff $f$ is differentiable on $[a, b]$ and, for every $\epsilon > 0$, there exists a $\delta > 0$ such that whenever $x_1, x_2 \in [a, b]$ with $|x_1 - x_2| < \delta$, it must follow that*

$$\left| \frac{f(x_1) - f(x_2)}{x_1 - x_2} - f'(x_1) \right| < \epsilon$$

**Corollary 28** *If $f : D \to \mathbb{R}$ is uniformly differentiable on $[a, b]$, then $f'$ is continuous on $[a, b]$.*

**Definition 7 (Lipschitz Condition)** *Let $f : D \to \mathbb{R}$. If there are positive constants $M$ and $\alpha$ such that for any $x_1, x_2 \in D$*

$$|f(x_1) - f(x_2)| \le M \cdot |x_1 - x_2|^{\alpha}$$

*then $f$ is* Lipschitz-$\alpha$ *with constant $M$, written $f \in \mathrm{Lip}_M\, \alpha$.*

**Theorem 29** *If $f \in \mathrm{Lip}_M\, \alpha$ on $D$, then*

1. *$f$ is continuous,*

2. *if* $\alpha > 1$, $f$ *is constant,*

**Corollary 30** *If* $f : [a, b] \to \mathbb{R}$ *is differentiable, then* $f \in \mathrm{Lip}_M\, 1$.

**Definition 8 (Measure Zero)** *A set* $E$ *has* measure zero *if and only if for any* $\epsilon > 0$ *the set* $E$ *can be covered by a countable collection of open intervals having total length less than* $\epsilon$; *i.e.,* $E \subseteq \bigcup_i (a_i, b_i)$ *where* $\sum_i (b_i - a_i) < \epsilon$.

**Definition 9 (Almost Everywhere)** *A property* $P$ *holds* almost everywhere *if the set* $\{x : P(x)$ *is not true.*$\}$ *has measure zero.*

**Theorem 31 (Rademacher's Theorem)** *If* $f \in \mathrm{Lip}_M\, 1$, *then* $f$ *is differentiable almost everywhere.*

**Theorem 32 (Lebesgue Differentiation Theorem)** *If* $f : \mathbb{R} \to \mathbb{R}$ *is continuous and nondecreasing, then* $f$ *is differentiable almost everywhere.*

**Definition 10 (Higher Order Derivatives)** *The* $n$*th derivative of* $f(x)$, *if it exists, is given by* $f^{(n)}(x) = \dfrac{d}{dx} f^{(n-1)}(x)$ *for* $n > 1$ *where* $f^{(0)} = f$.

**Theorem 33** *Let* $f : D \to \mathbb{R}$ *be* $m$ *times continuously differentiable. Then* $f$ *has* a root of multiplicity $m$ at $x = r$ *iff* $f^{(m)}(r) \neq 0$, *but*

$$f(r) = f'(r) = \cdots = f^{(m-1)}(r) = 0$$

**Theorem 34 (First Derivative Test for Extrema)** *Let* $f$ *be continuous on* $[a, b]$ *and differentiable on* $(a, b)$. *If* $f'(c) = 0$ *for* $c \in (a, b)$ *and* $f'$ *changes sign from*

- *negative to positive around* $c$, *then* $c$ *is a relative minimum of* $f$;
- *positive to negative around* $c$, *then* $c$ *is a relative maximum of* $f$.
- *If* $f'$ *does not change sign around* $c$, *then* $c$ *is a stationary "terrace point" of* $f$.

**Theorem 35 (Second Derivative Test for Extrema)** *Let* $f$ *be continuous on* $[a, b]$ *and twice differentiable on* $(a, b)$. *If* $f'(c) = 0$ *for* $c \in (a, b)$ *and*

- $f''(c)$ *is positive, then* $c$ *is a relative minimum of* $f$;
- $f''(c)$ *is negative, then* $c$ *is a relative maximum of* $f$;
- $f''(c) = 0$, *then the test fails.*

**Theorem 36 (Taylor's Theorem or Extended Law of the Mean)** *Let* $n \in \mathbb{N}$ *and suppose that* $f$ *has* $n + 1$ *derivatives on* $(a - h, a + h)$ *for some* $h > 0$. *Then for* $x \in (a - h, a + h)$

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!} (x - a)^k + R_n(x)$$

*where*

$$R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}\,(x-a)^{n+1}$$

*for some $c$ between $x$ and $a$.*

**Theorem 37 (Taylor's Theorem with Lagrange's Form of the Remainder)** *Let $n \in \mathbb{N}$ and suppose that $f$ has $n+1$ continuous derivatives on $(a-h, a+h)$ for some $h > 0$. Then for $x \in (a-h, a+h)$*

$$f(x) = f(a) + \sum_{k=1}^{n} \frac{f^{(k)}(a)}{k!}\,(x-a)^k + L_n(x)$$

*where*

$$L_n(x) = \frac{1}{n!}\int_a^x f^{(n+1)}(t)\cdot(x-t)^n\,dt.$$

**Theorem 38 (Bernstein)** *Let $I$ be an interval and $f : I \to \mathbb{R}$ have derivatives of all orders. If $f$ and all its derivatives are nonnegative, then the Taylor series of $f$ converges on $I$.*

**Theorem 39 (L'Hôpital's Rule)** *Suppose that $f$ and $g$ are differentiable on an open interval $I$ containing $a$, and that*

$$\lim_{x\to a} f(x) = 0 = \lim_{x\to a} g(x)$$

*while $g'(x) \neq 0$ on $I$. Then, if the limit exists,*

$$\lim_{x\to a} \frac{f'(x)}{g'(x)} = \lim_{x\to a} \frac{f(x)}{g(x)}$$

**Corollary 40** *Let $n \in \mathbb{N}$. Then*

$$\lim_{x\to\infty} \frac{x^n}{e^x} = 0 \quad \textit{and} \quad \lim_{x\to\infty} \frac{\ln(x)}{\sqrt[n]{x}} = 0$$

**Corollary 41** *If $f$ is twice differentiable on an open interval $I$ and $x \in I$, then*

$$f''(x) = \lim_{h\to 0} \frac{f(x+h) - 2f(x) + f(x-h)}{h^2}$$

## A.4 RIEMANN INTEGRATION

**Definition 11 (Partition)** *A* partition *$P$ of a closed interval $[a, b]$ is an ordered set of values $\{x_i \mid i = 0..n\}$ such that $a = x_0 < x_1 < \cdots < x_n = b$. The* norm *or* mesh *of the partition is*

$$\|P\| = \max\{\Delta x_i \mid i = 1..n\}$$

*where* $\Delta x_i = x_i - x_{i-1}$.

**Definition 12 (Cauchy Sum)** *Let* $f : [a,b] \to \mathbb{R}$ *be a continuous function and* $P$ *be a partition of* $[a,b]$. *Then the Cauchy sum (or "left endpoint sum") of* $f$ *(w.r.t.* $P$*) is*

$$C(P,f) = \sum_{k=1}^{n} f(x_{k-1})\Delta x_k$$

**Definition 13 (Riemann Sum)** *Let* $f : [a,b] \to \mathbb{R}$ *be a bounded function,* $P$ *be a partition of* $[a,b]$, *and* $c_k \in [x_{k-1}, x_k]$ *for each* $k$. *Then the Riemann sum of* $f$ *(w.r.t.* $P$ *and* $\{c_k\}$*) is*

$$R(P,f) = \sum_{k=1}^{n} f(c_k)\Delta x_k$$

**Definition 14 (Darboux Sums)** *Let* $f : [a,b] \to \mathbb{R}$, $P$ *be a partition of* $[a,b]$, *and set*

$$M_k(f) = \sup_{[x_{k-1},x_k]} f(x)$$
$$m_k(f) = \inf_{[x_{k-1},x_k]} f(x).$$

*Then the upper and lower Darboux sums of* $f$ *(w.r.t.* $P$*) are*

$$U(P,f) = \sum_{k=1}^{n} M_k \Delta x_k$$

$$L(P,f) = \sum_{k=1}^{n} m_k \Delta x_k,$$

*respectively.*

**Lemma 42** *Let* $f : [a,b] \to \mathbb{R}$ *be a bounded function, say, below by* $m$ *and above by* $M$, *and let* $P$ *be a partition of* $[a,b]$. *Then*

$$m(b-a) \le L(P,f) \le R(P,f) \le U(P,f) \le M(b-a)$$

*for all choices of* $\{c_k\}$.

**Lemma 43** *Suppose* $f : [a,b] \to \mathbb{R}$ *is a bounded function and* $P$ *and* $Q$ *are partitions of* $[a,b]$. *If* $P \subseteq Q$, *(i.e.,* $Q$ *is a* finer *partition), then*

1. $L(P,f) \le L(Q,f)$ *and* $U(Q,f) \le U(P,f)$.
2. $L(P,f) \le U(Q,f)$.

**Definition 15** *Set*

$$\overline{\int_a^b} f(x)\,dx = \inf_P U(P,f)$$

*and*

$$\underline{\int_a^b} f(x)\,dx = \sup_P L(P, f)$$

**Lemma 44** *Let $f : [a, b] \to \mathbb{R}$ be a bounded function and $P$ be a partition of $[a, b]$. Then*

$$L(P, f) \le \underline{\int_a^b} f(x)\,dx \le \overline{\int_a^b} f(x)\,dx \le U(P, f)$$

**Definition 16** *A bounded function $f : [a, b] \to \mathbb{R}$ is* Riemann integrable *on $[a, b]$ if and only if*

$$\underline{\int_a^b} f(x)\,dx = \overline{\int_a^b} f(x)\,dx = \int_a^b f(x)\,dx = A \in \mathbb{R}$$

*Set $\mathfrak{R}[a, b] = \{f \mid f$ is Riemann integrable on $[a, b]\}$.*

**Theorem 45** *The bounded function $f : [a, b] \to \mathbb{R}$ is Riemann integrable on $[a, b]$ if and only if for any $\epsilon > 0$ there is a partition $P$ such that $U(P, f) - L(P, f) < \epsilon$.*

**Theorem 46** *If $f$ is monotone on $[a, b]$, then $f \in \mathfrak{R}[a, b]$.*

**Theorem 47** *If $f$ is continuous on $[a, b]$, then $f \in \mathfrak{R}[a, b]$.*

**Theorem 48** *If $f, g \in \mathfrak{R}[a, b]$ and $c \in \mathbb{R}$, then*

1. *$f + g \in \mathfrak{R}[a, b]$ and $\displaystyle\int_a^b (f + g)(x)\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$.*
2. *$cf \in \mathfrak{R}[a, b]$ and $\displaystyle\int_a^b cf(x)\,dx = c\int_a^b f(x)\,dx$.*

**Theorem 49** *If $f, g \in \mathfrak{R}[a, b]$ and $f(x) \le g(x)$ on $[a, b]$, then $\displaystyle\int_a^b f(x)\,dx \le \int_a^b g(x)\,dx$.*

**Theorem 50** *If $f \in \mathfrak{R}[a, b]$, then $|f| \in \mathfrak{R}[a, b]$ and*

$$\left|\int_a^b f(x)\,dx\right| \le \int_a^b |f(x)|\,dx$$

**Theorem 51** *Let $f : [a, b] \to \mathbb{R}$ is bounded and $c \in (a, b)$. Then $f \in \mathfrak{R}[a, b]$ if and only if $f \in \mathfrak{R}[a, c]$ and $f \in \mathfrak{R}[c, b]$, and further, $\displaystyle\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$.*

**Theorem 52** *If $f \in \mathfrak{R}[a,b]$ and $g$ is continuous on $f([a,b])$, then $g \circ f \in \mathfrak{R}[a,b]$.*

**Corollary 53** *Let $f, g \in \mathfrak{R}[a,b]$ and $n \in \mathbb{N}$. Then*

1. $f^n \in \mathfrak{R}[a,b]$
2. $f \cdot g \in \mathfrak{R}[a,b]$.

**Lemma 54** *Let $f \in \mathfrak{R}[a,b]$ and $c, d \in [a,b]$. Then*

1. $\displaystyle\int_c^c f(x)\,dx = 0,$
2. $\displaystyle\int_c^d f(x)\,dx = -\int_d^c f(x)\,dx.$

**Theorem 55 (Bonnet's Theorem or First Mean Value Theorem for Integrals)** *Let $f$ be continuous on $[a,b]$ and $0 \le g \in \mathfrak{R}[a,b]$. Then there exists a value $c \in (a,b)$ such that*

$$\int_a^b f(x)\,g(x)\,dx = f(c)\int_a^b g(x)\,dx$$

**Theorem 56 (Second Mean Value Theorem for Integrals)** *If $f$ is a monotone function on $[a,b]$, then there exists a value $c \in (a,b)$ such that*

$$\int_a^b f(x)\,dx = f(a)(c-a) + f(b)(b-c)$$

**Theorem 57 (The Fundamental Theorem of Calculus)**

- *Define $F(x) = \displaystyle\int_a^x f(t)\,dt$. Then $F$ is continuous and, if $f$ is continuous at $x_0$, then $F'(x_0) = f(x_0)$*
- *If $F' = f$ on $[a,b]$, then $\displaystyle\int_a^b f(x)\,dx = F(b) - F(a)$*

**Theorem 58 (Midpoint Rule for Numerical Integration)** *Suppose $f : [a,b] \to \mathbb{R}$ is integrable and that $|f''(x)| \le M_2$ on $[a,b]$. For $n$ given, set $x_k = a + k(b-a)/n$ with $k = 0..n$ and set $\bar{y}_k = f\left((x_k + x_{k+1})/2\right)$ for $k = 0..n-1$. Then*

$$\int_a^b f(x)\,dx \approx \frac{b-a}{n}\left(\bar{y}_0 + \bar{y}_1 + \cdots + \bar{y}_{n-1}\right)$$

*with the absolute value of the error bounded by $\frac{1}{24}\,\frac{(b-a)^3}{n^2} M_2$.*

**Theorem 59 (Trapezoid Rule)** *Suppose $f : [a,b] \to \mathbb{R}$ is integrable and that $|f''(x)| \le M_2$ on $[a,b]$. For $n$ given, set $x_k = a + k(b-a)/n$ and set $y_k = f(x_k)$ with $k = 0..n$. Then*

$$\int_a^b f(x)\,dx \approx \frac{b-a}{2n}\left(y_0 + 2y_1 + 2y_2 + \cdots + 2y_{n-1} + y_n\right)$$

*with the absolute value of the error bounded by* $\frac{1}{12}\,\frac{(b-a)^3}{n^2}M_2$.

**Theorem 60 (Simpson's Rule)** *Suppose* $f : [a,b] \to \mathbb{R}$ *is integrable and that* $\left|f^{(4)}(x)\right| \le M_4$ *on* $[a,b]$. *For* $n$ *even, set* $x_k = a + k(b-a)/n$ *and set* $y_k = f(x_k)$ *with* $k = 0..n$. *Then*

$$\int_a^b f(x)\,dx \approx \frac{b-a}{3n}\,(y_0 + 4y_1 + 2y_2 + 4y_3 + 2y_4 + \cdots + 4y_{n-1} + y_n)$$

*with the absolute value of the error bounded by* $\frac{1}{180}\,\frac{(b-a)^5}{n^4}M_4$.

**Theorem 61 (Cauchy-Bunyakovsky-Schwarz Inequality)** *If* $f, g \in \mathfrak{R}[a,b]$, *then*

$$\left[\int_a^b f(x)\,g(x)\,dx\right]^2 \le \left[\int_a^b f^2(x)\,dx\right]\cdot\left[\int_a^b g^2(x)\,dx\right]$$

## A.5 RIEMANN-STIELTJES INTEGRATION

**Definition 17** *Let* $\alpha$ *be a monotonically increasing function on* $[a,b]$. *For any partition* $P$ *define* $\Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1})$.

**Definition 18 (Upper & Lower Riemann-Stieltjes Integrals)** *Let* $f$ *be a function that is bounded and* $\alpha$ *be monotonically increasing on* $[a,b]$. *For each partition* $P$, *define the* upper *and* lower Riemann-Stieltjes sums *by*

$$U(P,f,\alpha) = \sum_{k=1}^{n} M_k \Delta\alpha_k$$
$$L(P,f,\alpha) = \sum_{k=1}^{n} m_k \Delta\alpha_k$$

*Now, define the* upper *and* lower Riemann-Stieltjes integrals *as*

$$\overline{\int_a^b} f(x)\,d\alpha(x) = \inf_P U(P,f,\alpha)$$
$$\underline{\int_a^b} f(x)\,d\alpha(x) = \inf_P L(P,f,\alpha)$$

**Definition 19** *If* $\overline{\int} f\,d\alpha = \underline{\int} f\,d\alpha$, *then* $f$ *is* Riemann-Stieltjes integrable *and we write* $f \in \mathfrak{R}(\alpha)$.

**Theorem 62** *A function* $f$ *is Riemann-Stieltjes integrable on* $[a,b]$ *if and only if for every* $\epsilon > 0$ *there is a partition* $P$ *of* $[a,b]$ *such that*

$$U(P,f,\alpha) - L(P,f,\alpha) < \epsilon.$$

**Theorem 63** *If $f$ is continuous on $[a, b]$, then $f \in \mathfrak{R}(\alpha)$ on $[a, b]$.*

**Theorem 64** *If $f$ is monotonic on $[a, b]$ and $\alpha$ is continuous, then $f \in \mathfrak{R}(\alpha)$ on $[a, b]$.*

**Theorem 65** *If $f$ is bounded & has finitely many discontinuities on $[a, b]$ and $\alpha$ is continuous at each discontinuity of $f$, then $f \in \mathfrak{R}(\alpha)$.*

**Theorem 66** *If $f$ is bounded on $[a, b]$ and $f \in \mathfrak{R}(\alpha)$, then there exists $m$ and $M \in \mathbb{R}$ such that*

$$m(\alpha(b) - \alpha(a)) \leq \int_a^b f(x)\, d\alpha(x) \leq M(\alpha(b) - \alpha(a)).$$

**Theorem 67** *Let $f$ and $g \in \mathfrak{R}(\alpha)$ on $[a, b]$ and $c \in \mathbb{R}$. Then*

- $\int_a^b cf(x)\, d\alpha(x) = c \int_a^b f(x)\, d\alpha(x)$
- $\int_a^b f(x)\, d\, c\alpha(x) = c \int_a^b f(x)\, d\alpha(x)$
- $\int_a^b (f + g)(x)\, d\alpha(x) = \int_a^b f(x)\, d\alpha(x) + \int_a^b g(x)\, d\alpha(x)$
- $\int_a^b f(x)\, d(\alpha_1(x) + \alpha_2(x)) = \int_a^b f(x)\, d\alpha_1(x) + \int_a^b f(x)\, d\alpha_2(x)$
- $f \cdot g \in \mathfrak{R}(\alpha)$

**Theorem 68** *Let $f$ and $g \in \mathfrak{R}(\alpha)$ on $[a, b]$. If $f(x) \leq g(x)$, then $\displaystyle\int_a^b f(x)\, d\alpha(x) \leq \int_a^b g(x)\, d\alpha(x)$*

**Theorem 69** *If $f \in \mathfrak{R}(\alpha)$, then $\displaystyle\left|\int_a^b f(x)\, d\alpha(x)\right| \leq \int_a^b |f(x)|\, d\alpha(x)$*

**Definition 20** *Define the* Heaviside function *to be $U(x) = \begin{cases} 0 & x \leq 0 \\ 1 & x > 0 \end{cases}$*

**Theorem 70** *If $f$ is bounded on $[a, b]$ and continuous at $x_0 \in (a, b)$, then*

$$\int_a^b f(x)\, dU(x - x_0) = f(x_0)$$

**Theorem 71** *Let $c_n \geq 0$ with $\sum_n c_n$ converging and $\{x_n\}$ be a sequence of distinct points in $(a, b)$. Define*

$$\alpha(x) = \sum_{k=1}^{\infty} c_k U(x - x_k)$$

*Let $f$ be continuous on $[a, b]$. Then*

$$\int_a^b f(x)\, d\alpha(x) = \sum_{k=1}^{\infty} c_k f(x_k)$$

**Theorem 72** *If $f$ is bounded and $\alpha' \in \Re(\alpha)$, then $f \in \Re(\alpha)$ iff $f\alpha' \in \Re$ and*

$$\int_a^b f(x)\, d\alpha(x) = \int_a^b f(x)\alpha'(x)\, dx$$

**Theorem 73 (Hölder's Inequality)** *Let $f$ and $g$ be in $\Re(\alpha)$ and let $p, q > 0$ be such that $\frac{1}{p} + \frac{1}{q} = 1$. Then*

$$\left| \int_a^b f(x)g(x)\, d\alpha(x) \right| \leq \left[ \int_a^b |f(x)|^p d\alpha(x) \right]^{1/p} \left[ \int_a^b |g(x)|^q d\alpha(x) \right]^{1/q}$$

*If $p = 2$, this is called the* Cauchy-Bunyakovski-Schwarz Inequality.

**Theorem 74 (Minkowski's Inequality)** *Let $p > 1$ and let $f^p$ and $g^p$ be in $\Re(\alpha)$. Then*

$$\left[ \int_a^b [f(x) + g(x)]^p\, d\alpha(x) \right]^{1/p} \leq \left[ \int_a^b |f(x)|^p d\alpha(x) \right]^{1/p} + \left[ \int_a^b |g(x)|^p d\alpha(x) \right]^{1/p}$$

## A.6 SEQUENCES AND SERIES OF CONSTANTS

**Definition 21** *A* real-valued sequence *is a function $a : \mathbb{N} \to \mathbb{R}$. Sequence terms are denoted by $a(n) = a_n$.*

**Definition 22** *Let $f : \mathbb{N} \to \mathbb{N}$ be a strictly increasing function. The composition $a \circ f$ forms a* subsequence *and is denoted by $a(f(k)) = a_{n_k}$.*

**Definition 23** *A sequence* converges, *written $\lim_{n\to\infty} a_n = L$, iff for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that, if $n > N$, then $|a_n - L| < \epsilon$.*

**Theorem 75 (Corollary to the Heine-Borel Theorem)** *A bounded sequence has a convergent subsequence.*

**Definition 24 (Cauchy Sequence)** *A sequence $a_n$ is* Cauchy *iff, for every $\epsilon > 0$, there is an $N \in \mathbb{N}$ such that if $n, m > N$, then $|a_n - a_m| < \epsilon$.*

**Theorem 76** *A sequence is Cauchy iff the sequence converges.*

**Definition 25** *A sequence $\{a_n\}$ is*

- monotonically increasing *iff* $a_n \leq a_{n+1}$ *for all* $n$,
- monotonically decreasing *iff* $a_n \geq a_{n+1}$ *for all* $n$.

**Theorem 77** *If* $\{a_n\}$ *is monotonic, then* $\{a_n\}$ *converges iff it is bounded.*

**Definition 26** *Let* $\{a_n\}$ *be a sequence. Define*

- $\limsup_{n\to\infty} a_n = \lim_{n\to\infty}\left(\sup_{k\geq n} a_k\right)$ *which is also equal to* $\inf_{n\geq 0}\left(\sup_{k\geq n} a_k\right)$
- $\liminf_{n\to\infty} a_n = \lim_{n\to\infty}\left(\inf_{k\geq n} a_k\right)$ *which is also equal to* $\sup_{n\geq 0}\left(\inf_{k\geq n} a_k\right)$

**Theorem 78** *Let* $\{a_n\}$ *be a real-valued sequence. Then* $\lim_{n\to\infty} a_n = a$ *iff* $\liminf_{n\to\infty} a_n = a = \limsup_{n\to\infty} a_n$.

**Definition 27** *Let* $\{a_n\}$ *be a sequence. The associated* series *is* $s_n = \sum_{k=1}^{n} a_n$. *The terms* $s_n$ *are called the* partial sums *of the series. The series converges to* $s$, *written as* $s = \sum_{k=1}^{\infty} a_n$ *iff the sequence of partial sums* $\{s_n\}$ *converges to* $s$.

**Theorem 79** *If* $s_n = \sum_{k=1}^{\infty} a_n$ *converges, then* $\lim_{n\to\infty} a_n = 0$.

**Definition 28 (Cauchy Series)** *A series* $s_n = \sum_{k=1}^{\infty} a_n$ *is called* Cauchy *iff for every* $\epsilon > 0$, *there is an* $N \in \mathbb{N}$ *such that, if* $n \geq m > N$, *then* $\left|\sum_{k=m+1}^{n} a_k\right| < \epsilon$.

**Theorem 80** *Every Cauchy series in* $\mathbb{R}$ *converges.*

**Theorem 81 (The Comparison Test)** *Let* $\sum_n a_n$ *be a series.*

- *If* $|a_n| \leq c_n$ *for all* $n \geq N_0$ *and* $\sum c_n$ *converges, then* $\sum a_n$ *converges.*
- *If* $a_n \geq d_n > 0$ *for all* $n \geq N_0$ *and* $\sum d_n$ *diverges, then* $\sum a_n$ *diverges.*

**Theorem 82 (The Limit Comparison Test)** *Let* $\sum_n a_n$ *and* $\sum_n b_n$ *be positive series. If* $0 < \lim_{n\to\infty} a_n/b_n < \infty$, *then the series either both converge or both diverge.*

**Theorem 83 (Cauchy Condensation Test)** *Let* $\{a_n\}$ *be a nonnegative decreasing sequence. Then the series* $\sum_n a_n$ *converges if and only if* $\sum_n (2^n a_{2^n})$ *converges.*

**Theorem 84 (The Ratio Test, I)** *The series* $\sum_n a_n$ *converges if* $\limsup_n \left|\frac{a_{n+1}}{a_n}\right| < 1$ *and diverges if* $\left|\frac{a_{n+1}}{a_n}\right| > 1$ *for* $n \geq N_0$.

**Corollary 85 (The Ratio Test, II)** *For the series* $\sum_n a_n$*, define* $\rho = \lim_n \left|\frac{a_{n+1}}{a_n}\right|$. *Then,*

- $\rho < 1$*, the series converges,*
- $\rho > 1$*, the series diverges,*
- $\rho = 1$*, the test fails.*

**Theorem 86 (The Root Test)** *For the series* $\sum_n a_n$*, define* $\rho = \limsup_n \sqrt[n]{|a_n|}$. *Then,*

- $\rho < 1$*, the series converges,*
- $\rho > 1$*, the series diverges,*
- $\rho = 1$*, the test fails.*

**Theorem 87 (The Integral Test)** *Let* $f : [1, \infty) \to \mathbb{R}$ *be a continuous, positive, decreasing function such that* $f(n) = a_n$. *Then the improper integral* $\int_1^\infty f(x)\, dx$ *converges if and only if the series* $\sum_n a_n$ *converges.*

**Theorem 88 (Alternating Series Test)** *If* $a_n \geq a_{n+1} > 0$ *for all* $n \in \mathbb{N}$ *and* $\lim_{n\to\infty} a_n = 0$*, then* $\sum_n (-1)^n a_n$ *converges.*

## A.7 SEQUENCES AND SERIES OF FUNCTIONS

**Definition 29 (Pointwise Convergence)** *A sequence of functions* $f_n : E \to \mathbb{R}$ converges pointwise *on* $E$ *iff for each* $x \in E$*, the sequence* $\{f_n(x)\}$ *converges.*

**Definition 30 (Convergence in Mean)** *A sequence of integrable functions* $f_n : [a, b] \to \mathbb{R}$ converges in mean *to* $f$ *iff* $\lim_{n\to\infty} \left[\int_a^b [f_n(x) - f(x)]^2 dx\right]^{1/2} = 0.$

**Definition 31 (Uniform Convergence)** *A sequence of functions* $f_n : E \to \mathbb{R}$ converges uniformly *to* $f$ *on* $E$ *iff for every* $\epsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that whenever* $n > N$*, then* $|f_n(x) - f(x)| < \epsilon$ *for all* $x \in E$.

**Definition 32 (Cauchy Criterion for Uniform Convergence)** *A sequence of functions* $f_n : E \to \mathbb{R}$ *converges uniformly on* $E$ *iff for every* $\epsilon > 0$ *there is an* $N \in \mathbb{N}$ *such that whenever* $n, m > N$*, then* $|f_n(x) - f_m(x)| < \epsilon$ *for all* $x \in E$.

**Theorem 89** *Let $f_n : E \to \mathbb{R}$ be a sequence of functions converging uniformly to $f$ on $E$. If each $f_n$ is continuous on $E$, then $f$ is continuous on $E$.*

**Theorem 90** *Let $f_n : E \to \mathbb{R}$ be a sequence of functions converging uniformly to $f$ on $E$. Then $\lim\limits_{n\to\infty} \lim\limits_{x\to a} f_n(x) = \lim\limits_{x\to a} \lim\limits_{n\to\infty} f_n(x)$ for $x, a \in E$.*

**Theorem 91** *Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of integrable functions converging uniformly on $[a,b]$. Then*

$$\lim_{n\to\infty} \int_a^b f_n(x)\,dx = \int_a^b \lim_{n\to\infty} f_n(x)\,dx.$$

**Theorem 92 (Dini's Theorem)** *Let $f_n : [a,b] \to \mathbb{R}$ be a monotonic sequence of functions converging pointwise to $f$ on $[a,b]$ where $-\infty < a < b < \infty$. Then $f_n$ converges uniformly to $f$ on $[a,b]$.*

**Theorem 93** *Let $f_n : [a,b] \to \mathbb{R}$ be a sequence of continuously differentiable functions converging pointwise to $f$ on $[a,b]$. Then if $f_n'$ converges uniformly on $[a,b]$, then $f_n$ converges uniformly to $f$ and $f_n'$ converges uniformly to the continuous function $f'$ on $[a,b]$.*

**Theorem 94 (The Weierstrass $M$-test)** *Let $\sum_n f_n(x)$ be a series of functions all defined on $D \subseteq \mathbb{R}$. If there is a convergent series of constants $\sum_n M_n$ such that for each $n$ we have $|f_n(x)| \leq M_n$ for all $x \in D$, then $\sum_n f_n(x)$ converges both uniformly and absolutely on $D$.*

**Theorem 95 (Abel's Uniform Convergence Test)** *Let $f_n : D \to \mathbb{R}$ be a bounded, monotonically decreasing sequence of functions and $\sum_n a_n$ be a convergent series of constants. Then $\sum_n a_n f_n(x)$ converges uniformly on $D$.*