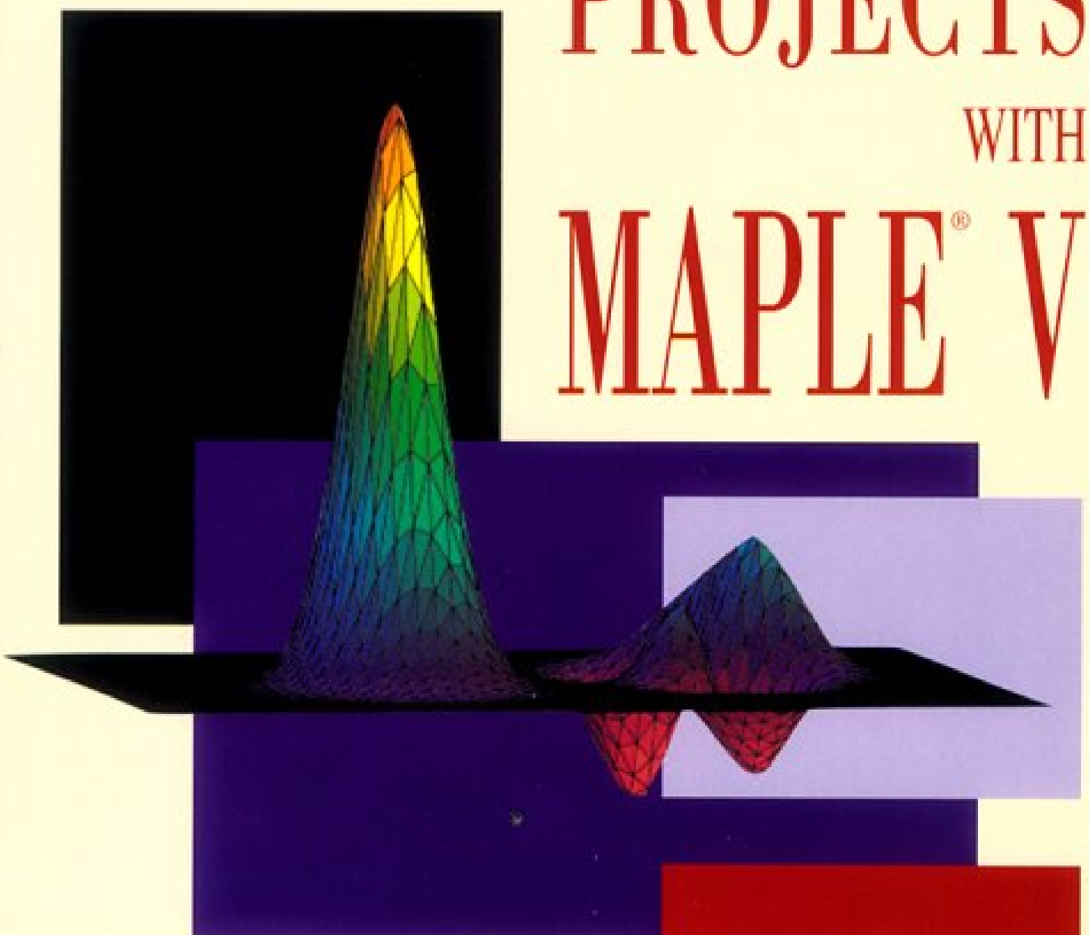
CALCULUS PROJECTS WITH MAPLE® V



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BROOKS/COLE SYMBOLIC COMPUTATION SERIES

Project 16

Taylor Polynomials and Convergence

> restart;

For this project you will need familiarity with the commands:

taylor	convert	seq
diff	plot	solve
fsolve		?

In addition, you will be using the *multiple plot* and *plot ranges* capability of the `plot` command and the *search interval* option of `fsolve`. You will also use the *polynom* option of `convert`.

Background

In this project and in the next several we seek computationally useful schemes of approximation. Since all calculations must ultimately reduce to addition and multiplication, it is natural to expect polynomials to play a special role. Thus, given the description of some function $f(x)$ of interest, we ask

> How may we choose the coefficients in
>
> $$T_n(x) := a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n$$
>
> to have $T_n(x)$ 'close' to $f(x)$?

Initially, assume the interval to be of the form $[-r, r]$ and fix our attention on the center of the interval, the point $x_0 = 0$. Then set $f(0) = T_n(0) = a_0$; since the functions go through the same point they have *order 0 contact* at $x_0 = 0$. A reasonable way of proceeding would be to require that $f'(0) = T_n'(0)$; matching derivatives gives *order* 1 *contact*. Since

$$T_n'(x) = a_1 + 2\,a_2\,x + 3\,a_3\,x^2 + \ldots + n\,a_n\,x^{n-1}$$

yields $f'(0) = T_n'(0) = a_1$, set $a_1 = f'(0)$. Proceeding once more, we require that $f''(0) = T_n''(0)$ giving *order 2 contact*. Then

$$T_n''(x) = 2\,a_2 + 3\cdot 2\,a_3\,x + 4\cdot 3\,a_4\,x^2 + \ldots n(n-1)\,a_n\,x^{n-2}$$

yields $f''(0) = T_n''(0) = a_2$ as our choice. Continuing the requirement that $f^{(k)}(0) = T_n^{(k)}(0)$ yields

$$a_k := \frac{f^{(k)}(0)}{k!}$$

Thus

$$T_n(x) := f(0) + f'(0)x + \frac{f''(0)}{2}x^2 + \frac{f'''(0)}{3!}x^3 + \ldots + \frac{f^{(n)}(0)}{n!}x^n$$

is called the nth Taylor[1] polynomial approximation (centered at zero) to f.

The natural and necessary question is are these polynomials any good? To answer that we will look at the error $R_n(x) := f(x) - T_n(x)$. The Mean Value Theorem can be employed to bound the remainder:

$$|R_n(x)| \leq \frac{1}{(n+1)!} \cdot M_{n+1} \cdot r^{n+1}$$

where $M_n + 1$ is any number that is greater than or equal to $|f^{(n+1)}(x)|$ for every $x \in [-r, r]$. Thus for $\sin(x)$, the Taylor polynomial of degree five

$$T_5(x) = x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

differs from the exact value of $\sin(x)$ by at most 0.00002 for every x in the interval $[-0.5, 0.5]$. Not bad for three terms.

Question 1 *Finish the computations to verify the claim that*

$$|R_5(x)| = |\sin(x) - T_5(x)| \leq 0.00002$$

for every x in the interval $[-0.5, 0.5]$.

Project Report

The Maple command `taylor(f(x), x, n+1)` returns $T_n(x)$, the nth Taylor polynomial for f, with an error term. Enter

> taylor(sin(x), x, 6);

$$x - \frac{1}{6}x^3 + \frac{1}{120}x^5 + \mathrm{O}(x^6)$$

The term "$O(x^6)$" represents the remainder, which we must remove. Enter

> T5 := convert(%, polynom);

[1]Named in recognition of the English mathematician Brooke Taylor, 1685–1731. Taylor presented the formula, now known as Taylor's Theorem in *Methodus Incrementorum Directa et Inversora*. The formula remained in oblivion for over 50 years due to his poor exposition.

$$T5 := x - \frac{1}{6}x^3 + \frac{1}{120}x^5$$

to remove the error term and convert the previous result to a polynomial. To see the potential of the method, produce a graph having domain $x = -8..8$ and range $y = -3..3$ showing $\sin(x)$ with $T_3(x)$, $T_5(x)$, and $T_{17}(x)$.

Question 2 *From your picture, give an interval on which the graph of $T_5(x)$ appears to coincide with the graph of $\sin(x)$ and an interval on which the graph of $T_{17}(x)$ appears to coincide with the graph of $\sin(x)$.*

Question 3 *Since $T_n(x)$ is a polynomial and the values of $\sin(x)$ lie between -1 and 1, can $T_n(x)$ ever be a good fit to $\sin(x)$ for all real values of x?*

Calculate the Taylor polynomials T_4, T_5, and T_{10} of $\cosh(x)$ centered at $x = 0$.

```
> T4 := convert(taylor(cosh(x), x=0, 5), polynom);
  T5 := convert(taylor(cosh(x), x=0, 6), polynom);
  T10 := convert(taylor(cosh(x), x=0, 11), polynom);
```

Question 4 *From plots of $\cosh(x)$, T_4, and T_{10} over the appropriate ranges:*

1. *In your opinion is T_{10} significantly better than T_4 as an approximation to $\cosh(x)$ for x between -0.5 and 0.5?*

2. *In your opinion is T_{10} is significantly better than T_4 as an approximation to $\cosh(x)$ for x between -5 and 5?*

3. *In your opinion is T_{10} is significantly better than T_4 as an approximation to $\cosh(x)$ for x between -15 and 15?*

Even though $T_n(x)$ gives a good approximation for a large class of functions, there are very simple functions for which T_n is inappropriate.

Question 5 *Why is it inappropriate to use a Taylor polynomial to approximate $f := x \to \arcsin(\sin(x))$ on the interval $[-\pi, \pi]$.*

We next proceed to quantify how well a Taylor polynomial T_k approximates a function $f(x)$. Operationally, the question becomes:

> Given an interval $[-r, r]$ and a maximum permissible error, ε, how large must k be chosen in order to guarantee
>
> $$|R_k(x)| = |f(x) - T_k(x)| < \varepsilon$$
>
> for every $x \in [-r, r]$?

As an example, we find a Taylor polynomial $T_n(x)$ that will approximate

$$f(x) = \sin(x) + \frac{1}{4}\sin(4x)$$

over the interval $[-3, +3]$ with a maximum error not exceeding 0.5. Recall

$$|R_n(x)| \leq \frac{M_{n+1}}{(n+1)!}\, r^{n+1}$$

where M_{n+1} is any number that is greater than or equal to $|f^{(n+1)}(x)|$ for every $x \in [-r, r]$. By replacing r with 3, we get

$$|R_n(x)| \leq \frac{3^{n+1}}{(n+1)!}\, M_{n+1}$$

Looking at successive derivatives of f, we see

```
> f := x -> sin(x) + sin(4*x)/4:
> diff(f(x), x);
```

$$\cos(x) + \cos(4x)$$

```
> diff(%, x);
```

$$-\sin(x) - 4\sin(4x)$$

```
> diff(%, x);
```

$$-\cos(x) - 16\cos(4x)$$

```
> diff(%, x);
```

$$\sin(x) + 64\sin(4x)$$

Since $|\cos(x)| \leq 1$ and $|\sin(x)| \leq 1$, the above computations yield

$$\begin{array}{rcccl} |f'(x)| & \leq & |\cos(x)| + |\cos(4x)| & \leq & 1+1, \\ |f''(x)| & \leq & |\sin(x)| + |4\sin(4x)| & \leq & 1+4, \\ |f^{(3)}(x)| & \leq & |\cos(x)| + |16\cos(4x)| & \leq & 1+16, \\ |f^{(4)}(x)| & \leq & |\sin(x)| + |64\sin(4x)| & \leq & 1+64. \end{array}$$

Evidently, repeated use of the chain rule gives us

$$|f^{(n+1)}(x)| \leq 1 + 4^n,$$

so we choose $M_{n+1} = 1+4^n$. (Note that we have not *proved* the bound; that could be done with an Induction argument.) Next solve the inequality:

$$\frac{3^{n+1}}{(n+1)!} M_{n+1} = \frac{3^{n+1}}{(n+1)!} (1+4^n) \leq \frac{1}{2}$$

for n. Unfortunately, Maple has difficulties with this inequality:

```
> solve(3^(n+1)*4^(n+1)/(n+1)! ≤ 1/2, n);
```

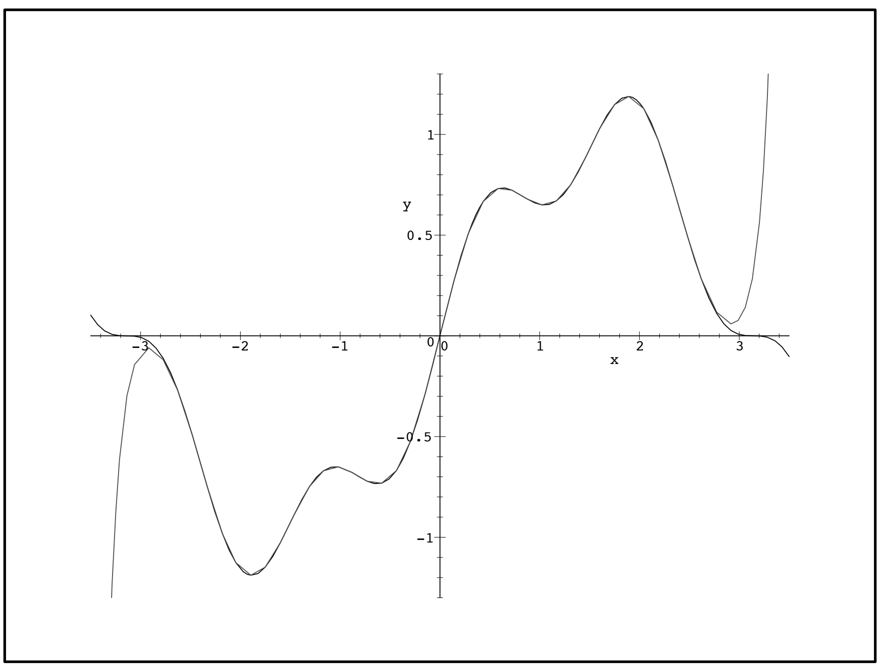
A null output[2] indicates that Maple could not find a solution. Looking at a plot is one way around this problem. However the authors have their own pet ways. One likes:

```
> fsolve(3^(n+1)*(1+4^n)/(n+1)! - 1/2, n=1..50);
```

The other prefers the brute force:

```
> seq([n, evalf(3^(n+1)*(1+4^n)/(n+1)!)], n=1..50);
```

Either of these two approaches yields $T_{30}(x)$ as the desired polynomial. In the graph below, we can observe that while $T_{30}(x)$ is a good approximation to $f(x)$ between -3 and 3, $T_{30}(3.5)$ is already off the screen.



1. Find the degree, n, necessary for $T_n(x)$ to approximate $y = \cosh(x) + \cosh(2x)$ over the interval $[-3, +3]$ with a maximum error not exceeding 0.5. Plot $y = \cosh(x) + \cosh(2x)$ and $T_n(x)$ on the same graph.

[2] Users may see an error message: `unable to determine sign of expression`.

Extension

Find the "best" center a to generate a Taylor polynomial of degree n, specified by your instructor, to approximate $g := x \to e^x$ over the interval $[-2, 2]$.

Report Requirements

A minimal project report will include:

- English responses and accompanying plots for Questions 1 through 5.
- An explication of the computations and plots required in Problem 1.