COMPUTER ALGEBRA AND INTEGRATION

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QUESTIONS

- How does computer algebra affect the learning of integration?

- What techniques are still important?

- Which other methods are unnecessary in first-year instruction?

PATTERNS, I.

Consider the four integrals:

$$\int \frac{1}{1+x^2} = \arctan(x)$$

$$\int \cos(x) = \sin(x)$$

$$\int \frac{1}{\sqrt{1-x^2}} = \arcsin(x)$$

$$\int \operatorname{arccosh}(x) = x \operatorname{arccosh}(x) - \sqrt{x^2-1}$$

(from Geddes, Szapor, and Labahn *Algorithms for Computer Algebra*)

PATTERNS, II.

The four integrals represented with logarithms and exponentials:

$$\int \frac{1}{1+x^2} = \frac{1}{2}\left(\log(x+\imath) - \log(x-\imath)\right)\cdot\imath$$

$$\begin{aligned}\int \cos(x) &= \tfrac{1}{2}\int\left(e^{\imath x}+e^{-\imath x}\right)\\ &= \tfrac{\imath}{2}\left(-e^{\imath x}+e^{-\imath x}\right)\end{aligned}$$

$$\int \frac{1}{\sqrt{1-x^2}} = -\imath\,\log\left(\imath x+\sqrt{1-x^2}\right)$$

$$\begin{aligned}\int \operatorname{arccosh}(x) &= \int \log\left(x+\sqrt{x^2-1}\right)\\ &= x\,\log\left(x+\sqrt{x^2-1}\right)-\sqrt{x^2-1}\end{aligned}$$

The Integration Problem

Given two class of functions, $\mathcal{F}$ and $\mathcal{G}$, find an algorithm that, for every $f \in \mathcal{F}$, presents a $g \in \mathcal{G}$ such that $g' = f$ or proves no such g exists.

DEFINITIONS

Algebraic: θ is *algebraic* over $\mathcal{K}$ if θ satifies a polynomial with coefficients from $\mathcal{K}$.

Logarithmic: θ is *logarithmic* over $\mathcal{K}$ if there is a $\lambda \in \mathcal{K}$ such that $\theta'\lambda = \lambda'$.

Exponential: θ is *exponential* over $\mathcal{K}$ if there is a $\eta \in \mathcal{K}$ such that $\theta' = \eta'\theta$.

I. LIOUVILLE – 1832

Integration in Finite Terms – *Liouville's Principle*
Let f be from a function field $\mathcal{K}$. If f has an elementary integral over $\mathcal{K}$, then it is of the form

$$\int f = v_0 + \sum c_i \log(v_i)$$

where $v_0 \in \mathcal{K}$, $v_i \in \mathcal{K}^*$, a finite algebraic extension of $\mathcal{K}$, and the constants $c_i \in \mathcal{K}^*$.

II. HERMITE – 1872

Rational Functions – Algorithm (Partial Fractions)

III. RISCH – 1968

Logarithmic and Exponential Extension Fields – Extended Algorithm
Add finitely many transcendental extensions, then decompose the integral as in Hermite's method

IV. DAVENPORT/BRONSTEIN – 1988/1992

Transcendental/Algebraic Extension – Extends Risch's Algorithm

LIOUVILLE'S PRINCIPLE – 1832

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RISCH'S THEOREM – 1968

Let $\mathcal{C}$ be a field of constants. Let $\mathcal{K} = \mathcal{C}(x, \theta_1, \ldots, \theta_n)$ be a field of functions with θ_i being either an exponential or a logarithm of an element in $\mathcal{C}(x, \theta_1, \ldots, \theta_{i-1})$ for $i = 1..n$ and, hence, $\mathcal{C}(x, \theta_1, \ldots, \theta_i)$ is a transcendental extension of $\mathcal{C}(x, \theta_1, \ldots, \theta_{i-1})$. The field of constants of $\mathcal{K}$ is $\mathcal{C}$. Then there is an algorithm that, for $f \in \mathcal{K}$, either gives an elementary function g over $\mathcal{K}$ such that $\int f = g$ or proves that f has no elementary integral over $\mathcal{K}$.

DAVENPORT'S EXTENSION – 1988

Replace $\mathcal{C}(x, \theta_1, \ldots, \theta_n)$ with $\mathcal{C}(x, y, \theta_1, \ldots, \theta_n)$ where y is algebraic over $\mathcal{C}(x)$.

EXAMPLE

Consider $\int e^{x^2}$.

If $\int e^{x^2}$ has an elementary integral, then, by Liouville's Principle, the integral must have the form

$$\int e^{x^2} = R(x) \cdot e^{x^2}$$

where $R = N/D$ is rational and has $\gcd(N, D) = 1$. Differentiate both sides:

$$e^{x^2} = R' \cdot e^{x^2} + 2xR \cdot e^{x^2}$$

So

$$\begin{aligned} 1 &= R' + 2xR \\ D^2 &= N'D - D'N + 2xND \end{aligned}$$

Rearrange to obtain $D \cdot (D - N' - 2xN) = -D'N$. A simple argument based on roots of D shows no such R exists. Therefore $\int e^{x^2}$ has no elementary integral.

HERMITE'S METHOD

I. Square-Free Factorization

$$\frac{p}{q} = \frac{p}{\prod_{i=1}^{n} q_i^i} = \sum_{i=1}^{n} \frac{p_i}{q_i^i}, \qquad \deg(p_i) \leq i - 1$$

II. Exponent Reduction

$$\begin{aligned} \int \frac{p_i}{q_i^i} &= \int \frac{p_i \cdot (a q_i + b q_i')}{q_i^i} \\ &= \int \frac{p_i a}{q_i^{i-1}} + \int \frac{p_i b q_i'}{q_i^i} \\ &= -\frac{p_i b/(i-1)}{q_i^{i-1}} + \int \frac{p_i a + (p_i b)'/(i-1)}{q_i^{i-1}} \end{aligned}$$

III. Synthesis

$$\int \frac{p}{q} = \sum_{i=1}^{n} \left(c_{i,0} \log(q_i) + \sum_{i=1}^{i-1} \frac{c_{i,j}}{q_i^j} \right)$$

HOROWITZ' METHOD

I. Separate Logarithmic Integral

$$\begin{aligned}\int \frac{q}{r} &= \frac{q_1}{r_1} + \int \frac{q_2}{r_2} \\ &= \frac{q_1}{r_1} + \sum_{k=1}^{j} c_k \log(r_{2,k})\end{aligned}$$

II. Logarithmic Sum Denominator

$$r_2 = r / \gcd(r, r')$$

TRAGER / ROTHSTEIN IMPROVEMENT

I. Logarithmic Part

Suppose that $\int \frac{p}{q} = \sum_{k=1}^{n} c_k \log(v_k)$. Then

$$c_k \in \text{RootOf}\{\text{Res}_x(p - y\,q', q)\}$$

and

$$v_k = \gcd(p - c_k\,q', q)$$

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