MAT 5930. Analysis for Teachers.

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Course Outline

1. Introduction
2. Calculus Review
   Week 1
   - Calculus Courses
     - Standard Courses
     - AP Calculus Courses
   - Calculus Problems
     - §1 Precalculus Background
     - §2 Limits & Continuity
     - §3 Derivatives
     - §4 Integration
     - §5 Infinite Series
3. Analysis Problems
   Weeks 2-3
   - §1 Basic Problems
   - §2 Supplementary Problems
   - §3 Enrichment Problems
4. History & Biography
   Week 4
5. Readings
6. Student Presentations & Reports
   Week 4

Introduction and Calculus Review

1. Course Introduction (Course Info page, Syllabus, Projects)

2. Calculus Review

   2.1 A Standard Freshman Calculus Course (§I–III)

   - Refer to texts by *Thomas*, *Stewart*, and *Ostebee & Zorn*

   2.2 An AP Calculus Course

   2.2.1 Functions, Graphs, and Limits

   Analysis of graphs, limits of functions, asymptotic behavior, continuity

   2.2.2 Derivatives

   Concept, interpretations, at a point, as a function, second derivative, applications, computation, numerical approximation

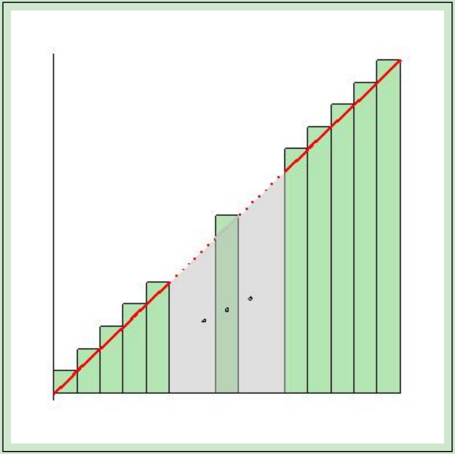
   2.2.3 Integrals

   Concept, interpretations, properties, Fundamental Theorem, applications, techniques, applications, numerical approximation

2. (Calculus Review)
   2.3 Calculus Problems
   - §1 **Precalculus material**: *summation, induction, slope, trigonometry*; Pg. 7, 1–4.
   - §2 **Limits and Continuity**: *Squeeze Theorem, discontinuity, removable discontinuity, different interpretations of limit expressions*; Pg. 9, 5–7.
   - §3 **Derivatives**: *trigonometric derivatives, power rule, indirect methods, Newton's method, Mean Value Theorem, "Racetrack Principle"*; Pg. 10, 8–12.
   - §4 **Integration**: *Fundamental Theorem, Riemann sums, parts, multiple integrals*; Pg. 12, 13–20.
   - §5 **Infinite Series**: *geometric, integrals and series, partial fractions, convergence tests (ratio, root, comparison, integral), Taylor & Maclaurin*; Pg. 14, 21–25.

Summations and Induction

Example ($\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ by Picture)



Example ($\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$ by Induction)

Let $P(n)$ be the proposition that $\sum_{k=1}^{n} k$ is equal to $\frac{n(n+1)}{2}$.

Basis: Then $P(1)$, which is $\sum_{k=1}^{1} k = \frac{1(1+1)}{2}$, is clearly true.

Induction: Show that if $P(n)$ is true, then $P(n+1)$ is true.
Assume $P(n)$ is true and add $(n+1)$ to both sides; i.e.,

$$(n+1) + \sum_{k=1}^{n} k = (n+1) + \frac{n(n+1)}{2}.$$

Combine terms to see $\sum_{k=1}^{n+1} k = \frac{(2n+2)+n(n+1)}{2}$. Simplify:

$$\sum_{k=1}^{n+1} k = \frac{(n+1)(n+2)}{2}$$

which shows that $P(n+1)$ is true.

By the *Principle of Mathematical Induction*, the result holds.

Summations and Induction

Exercise

Validate the formula by picture and prove it by induction:

1. $\sum_{k=1}^{n} 2k = n^2 + n$
2. $\sum_{k=1}^{n} 2k - 1 = ?$ (HINT: A STACK OF TRIANGLES)
3. $\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}$

Compare:

Definition (Intuitive Limit)

The *limit of* $f(x)$ *as* x *approaches* a *is* L, written as

$$\lim_{x \to a} f(x) = L,$$

if and only if we can make $f(x)$ *arbitrarily close to* L whenever x is *sufficiently close to*, but not equal to, a.

Definition (Formal Limit—*calculus level*)

Let f be defined on an open interval $\mathcal{I}$ containing a, but not necessarily defined at a. Then

$$\lim_{x \to a} f(x) = L$$

if and only if for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that whenever $x \in \mathcal{I}$ $(x \neq a)$ is within δ of a, then $f(x)$ must be within ϵ of L.

Limit Proofs, I

Example (ϵ–δ Proof)

Prove: $\lim_{x \to 2} 2x + 3 = 7$

Proof.

Let $\epsilon > 0$. Then we need to find a $\delta > 0$ so that

$$\begin{aligned} |f(x) - L| = |(2x+3) - (7)| &< \epsilon \\ |2x - 4| &< \epsilon \\ 2\,|x - 2| &< \epsilon \end{aligned}$$

Choosing $\delta > 0$ to be less than $\epsilon/2$ yields that if $0 < |x - 2| < \delta$, then it must follow that $|f(x) - L| = |(2x+3) - (7)| < \epsilon$. □

Limit Proofs, II

Example (ϵ–δ Proof)

Prove: $\lim_{x \to 3} 4x^2 - 1 = 35$

Proof.

Let $\epsilon > 0$. Then we need to find a $\delta > 0$ so that

$$\begin{aligned} |f(x) - L| = \left|(4x^2 - 1) - (35)\right| &< \epsilon \\ |4x^2 - 36| = |2x + 6| \cdot |2x - 6| &< \epsilon \\ (4\,|x + 3|) \cdot |x - 3| &< \epsilon \end{aligned}$$

Assume $\delta < 1$. Then $-1 < x - 3 < 1$ implies that $5 < x + 3 < 7$, so that $20 < 4(x + 3) < 35$. Choosing $\delta > 0$ to be less than the minimum of $\epsilon/35$ and 1 yields that if $0 < |x - 3| < \delta$, then it must follow that $|f(x) - L| = \left|(4x^2 - 1) - (35)\right| < \epsilon$. □

Limit Proofs, III

"For every $\epsilon > 0$ there is a $\delta > 0$ such that P is true"

negated becomes

"There is an $\epsilon > 0$ for which no $\delta > 0$ gives that P is true"

Example (ϵ–δ "Non-Proof")

Demonstrate that $\lim_{x \to 0} \frac{|x|}{x}$ does not exist.

Proof.

Suppose the limit is L. Let $\epsilon = 1$ and let δ be any positive number. Choose any $x_p \in (0, \delta)$. Then $|x_p|/x_p = 1$, so that $|f(x) - L| = |1 - L|$. Choose any $x_n \in (-\delta, 0)$. Then $|x_n|/x_n = -1$, so that $|f(x) - L| = |-1 - L|$. We have that

$$-\epsilon < 1 - L < \epsilon \;\Rightarrow\; -1 < 1 - L < 1 \;\Rightarrow\; 0 < L < 2$$

and

$$-\epsilon < -1 - L < \epsilon \;\Rightarrow\; -1 < -1 - L < 1 \;\Rightarrow\; -2 < L < 0$$

which is a contradiction. Therefore there is no limit L. □

Limit Proofs, IV

Exercise

Find the value and prove it correct for:

1. $\lim_{x \to 3} 4x - 1 =$
2. $\lim_{x \to 2} -3x + 5 =$
3. $\lim_{x \to 1} 4 - x^2 =$
4. $\lim_{x \to 0} 2x^3 + 1 =$
5. $\lim_{x \to -1} 3x^3 + x + 1 =$ (SEE A PROOF)
6. *Why won't this approach work for* $\lim_{x \to 1} \ln(2x - 1)$ *?*
7. *Prove that* $\lim_{x \to 0} \frac{1}{x}$ *doesn't exist.*

Compare:

Definition (Intuitive Continuity)

The function f is *continuous* at $x = a$ if and only if we can make $f(x)$ *arbitrarily close to* $f(a)$ whenever x is *sufficiently close to* a.

Definition (Formal Continuity—*calculus level*)

Let f be defined on an open interval $\mathcal{I}$ containing a. Then f is *continuous* at $x = a$ if and only if for every $\epsilon > 0$ there is a corresponding $\delta > 0$ such that whenever $x \in \mathcal{I}$ is within δ of a, then $f(x)$ must be within ϵ of L.

1. How do these definitions compare to the limit definitions?

Squeeze Theorem

Theorem (Squeeze Theorem or Sandwich Theorem)

Suppose that $m(x) \leq f(x) \leq M(x)$ *on a deleted neighborhood*[1] *of* a *and that*

$$\lim_{x \to a} m(x) = L = \lim_{x \to a} M(x).$$

Then

$$\lim_{x \to a} f(x) = L.$$

1. Apply the theorem to $f(x) = x^2 \sin(1/x)$ to determine a value for $f(0)$ that makes f continuous.
2. State and apply an analogue of the theorem to use to determine $\lim_{x \to \infty} \dfrac{\sin(x)}{x}$.

[1] A deleted neighborhood of a is $(a - \delta, a) \cup (a + \delta)$ for some $\delta > 0$.

Types of Discontinuity

Definition (Four Principal Types of Discontinuity)

Removable: The limit $\lim_{x\to a} f(x)$ exists, but isn't equal to $f(a)$.

Jump: Both $\lim_{x\to a+} f(x)$ and $\lim_{x\to a-} f(x)$ exist, but have different values.

Infinite: At least one of $\lim_{x\to a+} f(x)$ or $\lim_{x\to a-} f(x)$ is infinite.

Oscillating: At least one of $\lim_{x\to a+} f(x)$ or $\lim_{x\to a-} f(x)$ doesn't exist, but is bounded.

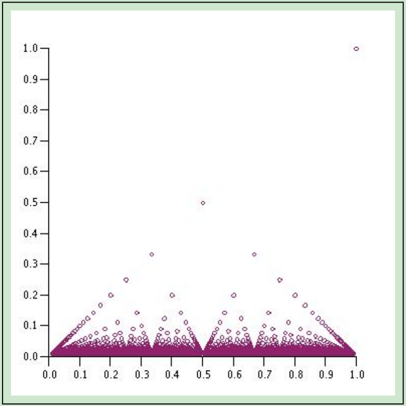
1. Find examples of each type of discontinuity.

Continuous and Discontinuous *a Lot*!

Example (Dirichlet's Function)

A function that is continuous at each irrational point, discontinuous at each nonzero rational point in $[0, 1]$.

$$D(x) = \begin{cases} 1/q & \text{if } x = p/q \\ 0 & \text{otherwise} \end{cases}$$



Derivatives

Definition (The Derivative Function)

The *derivative* of $f(x)$ is given by

$$\frac{d}{dx} f(x) = f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

if the limit exists.

Definition (Rules)

Results:

1. $(u^r)' = r\, u^{r-1} \cdot u'$
2. $(e^u)' = e^u \cdot u', \ln(u)' = \frac{u'}{u}$
3. $\sin(u)' = +\cos(u) \cdot u'$, etc.
4. $\sin^{-1}(u)' = \frac{u'}{\sqrt{1-u^2}}$, etc.

Reductions:

1. $(k \cdot u)' = k \cdot (u')$
2. $(u \pm v)' = (u') \pm (v')$
3. $(u \cdot v)' = (u') \cdot v + u \cdot (v')$
4. $\left(\frac{u}{v}\right)' = \frac{(u') \cdot v - u \cdot (v')}{v^2}$
5. $(u(v))' = u'(v) \cdot v'$

Newton's Method

Example (A Functional Version of Newton's Method)

Define the function $\mathrm{Newton}(x) = x - \dfrac{f(x)}{f'(x)}$.

Maple	TI
`f := x -> ...;`	`y1 := ...`
`df := D(f);`	`y2 := (y1′)`
`N := x -> x - f(x)/df(x);`	`y3 := x - y1(x)/y2(x)`

Give an initial value and iterate:

Maple	TI
`1.0;`	`1.0`
`N(%);`	`y3(ans)`
`N(%);`	`y3(ans)`

Et cetera.

1. Find all positive roots of $f(x) = x^7 - 1.4995x + 0.994$

The Mean Value Theorem and . . .

Theorem (The Mean Value Theorem)

Let f *be differentiable on* (a, b) *and continuous at the endpoints. Set* $m = \dfrac{f(b) - f(a)}{b - a}$*. Then there is a* $c \in (a, b)$ *so that* $f'(c) = m$.

Theorem (The "Speed Limit Law")

Let f *be differentiable on* $[a, b]$ *and* $M \in \mathbb{R}$*. If* $f'(x) \leq M$ *for all* $x \in [a, b]$*, then* $f(b) - f(a) \leq M \cdot (b - a)$.

Theorem (The "Racetrack Principle")

Suppose $f(a) = g(a)$ *and* $f'(x) \leq g'(x)$ *for all* $x \geq a$*. Then* $f(x) \leq g(x)$ *for all* $x \geq a$.

Fundamental Theorem of Calculus

Theorem (Fundamental Theorem, Version 1)

Suppose that f *is integrable on* $[a, b]$ *and set*

$$F(x) = \int_a^x f(t)\,dt$$

for $x \in [a, b]$. *Then* F *is continuous and at each point of continuity* c *of* f *we have that* $F'(c) = f(c)$.

Theorem (Fundamental Theorem, Version 2)

Let f *be continuous on* $[a, b]$ *with* $F'(x) = f(x)$. *Then*

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

Functions Defined by Integrals

Many important functions that have no elementary expressions are defined by integrals.

Definition (Several Special Functions)

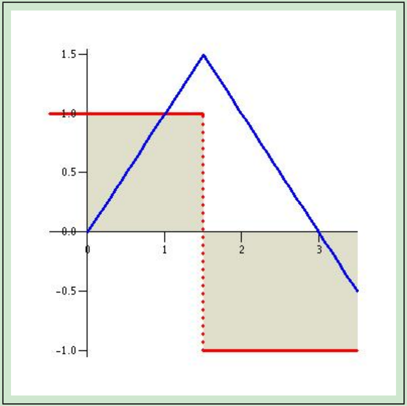
- $\ln(x) = \int_1^x \frac{1}{t}\,dt$
- $\operatorname{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\,dt$
- $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\,dt$ *Generalized factorial as* $\Gamma(n) = (n-1)!$
- $\operatorname{Si}(x) = \int_0^x \frac{\sin(t)}{t}\,dt$
- $\mathcal{F}_s(x) = \int_0^x \sin(\pi/2 \cdot t^2)\,dt$

and hundreds more . . .

A Function Defined by an Integral

Example

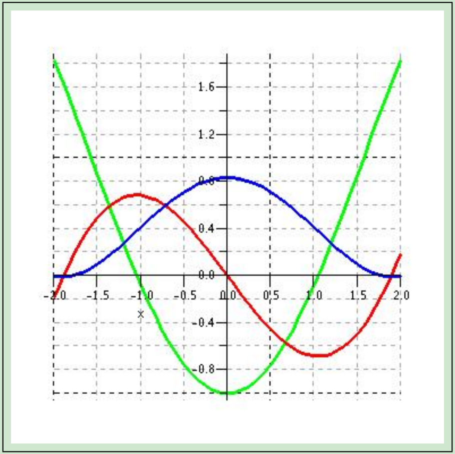
Define: $f(x) = \begin{Bmatrix} 1 & x < 1.5 \\ -1 & \text{otherwise} \end{Bmatrix}$ and $F(x) = \int_0^x f(t)\,dt$.



Query: Where is F not differentiable?

Which Witch is Which?

Example



The graph shows $f(x)$, $f'(x)$, and $\int f(x)\,dx$. Which is which?

Riemann Sums

Definition (Riemann Sum)

Let f be a bounded function on $[a, b]$. Let the *partition* $\mathcal{P}$ be $\mathcal{P} = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$ and $\mathcal{T} = \{t_i\}$ be a collection of points where $t_i \in [x_{i-1}, x_i]$ for each $i = 1..n$. The Riemann sum is

$$\mathcal{R}(f, \mathcal{P}, \mathcal{T}) = \sum_{k=1}^{n} f(t_k) \cdot (x_k - x_{k-1})$$

Definition (Riemann-Stieltjes Sum)

Let f be bounded and g be increasing on $[a, b]$. Let the *partition* $\mathcal{P}$ be $\mathcal{P} = \{x_0 = a < x_1 < x_2 < \cdots < x_n = b\}$ and $\mathcal{T} = \{t_i\}$ be a collection of points where $t_i \in [x_{i-1}, x_i]$ for each $i = 1..n$. The Riemann-Stieltjes sum is

$$\mathcal{RS}(f, g, \mathcal{P}, \mathcal{T}) = \sum_{k=1}^{n} f(t_k) \cdot [g(x_k) - g(x_{k-1})]$$

A Special Integral

The integral $\mathcal{I} = \displaystyle\int_0^\infty e^{-(x^2)}dx$ is important in analysis and probability, but has no elementary antiderivative — the Fundamental Theorem does not apply. Instead, consider

$$\mathcal{I}^2 = \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy.$$

Change to *polar coordinates* with

$[x, y] \mapsto [r, \theta] = [\sqrt{x^2 + y^2}, \arctan(y/x)]$ and $dx\, dy \mapsto r\, dr\, d\theta.$

The transformed integral is

$$\mathcal{I}^2 = \int_0^{\pi/2} \int_0^\infty e^{-r^2} r\, dr\, d\theta$$

which is not hard to evaluate.

Convergence Tests

Theorem

Ratio Test: *Let* $\sum a_n$ *be a series of positive terms and set* $r = \lim a_{n+1}/a_n$. *Then*

1. *if* $0 \leq r < 1$, *the series converges.*
2. *if* $1 < r \leq \infty$, *the series diverges.*
3. *if* $r = 1$, *the test fails.*

Root Test: *Let* $\sum a_n$ *be a series of positive terms and set* $\rho = \lim \sqrt[n]{a_n}$. *Then*

1. *if* $0 \leq \rho < 1$, *the series converges.*
2. *if* $1 < \rho \leq \infty$, *the series diverges.*
3. *if* $\rho = 1$, *the test fails.*

Further Tests

Theorem

Limit Comparison: *Let* $\sum a_n$ *and* $\sum b_n$ *be positive series. Set* $r = \lim a_n/b_n$. *Then*

1. *if* $r = 0$ *and* $\sum b_n$ *converges, then* $\sum a_n$ *converges.*
2. *if* $0 < r < \infty$, *the series either converge or diverge together.*
3. *if* $r = \infty$ *and* $\sum b_n$ *diverges, then* $\sum a_n$ *diverges.*

Integral Test: *Let* $a_n = f(n)$ *be positive terms. Then* $\sum a_n$ *and* $\int_k^\infty f(x)\,dx$ *converge or diverge together.*

For more convergence tests, visit *Mathworld*.

Taylor Polynomials and Series

Definition (Taylor Polynomial for f)

Let f have n continuous derivatives. The Taylor polynomial for f of degree n is

$$T_n(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \cdots + \frac{f^{(n)}(a)}{n!}(x-a)^n$$

Set $M_{n+1} = \max |f^{(n+1)}(t)|$ on the interval $[a, b]$. Then the error in approximating f by T_n for $x \in [a, b]$ is bounded by

$$\mathrm{Err}_n \leq \frac{M_{n+1}}{(n+1)!}(x-a)^{n+1}$$

Definition (Taylor Series for f)

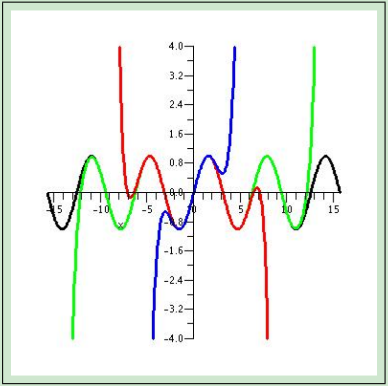
Let f have derivatives of all orders. The Taylor series for f is

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$$

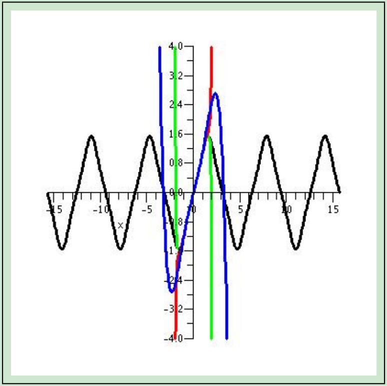
for $|x - a| < R$ where $R \geq 0$ is the *radius of convergence*.

Taylor Comparison

Example



$\sin(x)$ & T_5, T_{15}, and T_{30}.

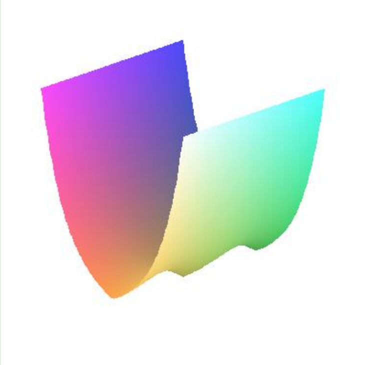


$\tan(\sin(x))$ & T_5, T_{15}, and T_{30}

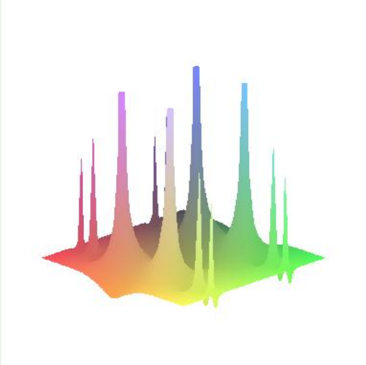
The Taylor series for sine works well, in contrast to $\tan \circ \sin$'s.

Complex Plots Show the Answer

Example



$z = |\sin(x + iy)|$



$z = |\tan(\sin(x + iy))|$

The *radius of convergence* of a Taylor series is the distance from its center to the nearest pole. Poles may lie in the plane, off the real axis.

3. (Analysis Problems)
   3.3 Enrichment Problems
   - The Rationals Are a Small Set
     Ex. 1-2, pg. 100.
   - A Brief Introduction to Lebesgue Measure
     Ex. 2-4, pg. 122.
   - Special Functions — the Gamma Function
     Ex. 1-5, pg. 108.
   - Fourier Series
     Ex. 1-5, pg. 115.

Even & Odd Functions

Theorem

Every function $f : \mathbb{R} \to \mathbb{R}$ is the sum of an even function and an odd function.

Proof.

Given a function $f : \mathbb{R} \to \mathbb{R}$, define the two components $f_e(x)$ and $f_o(x)$ as

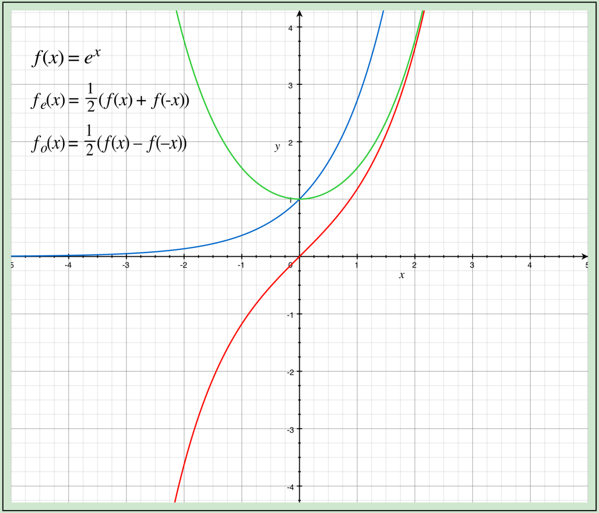
$$f_e(x) = \frac{1}{2}\left(f(x) + f(-x)\right)$$
$$f_o(x) = \frac{1}{2}\left(f(x) - f(-x)\right)$$

□

Exercise: Fill in the details to make this a proof.

Even & Odd Decomposition

Example



Even Function Integrals

Example

Integrate $\displaystyle\int_{-\pi}^{+\pi} \sin^2(x)\,dx$.

The function $\sin^2$ is even. Thus

$$\int_{-\pi}^{+\pi} \sin^2(x)\,dx = 2\int_{0}^{+\pi} \sin^2(x)\,dx$$

Apply a trigonometric identity to see that

$$\begin{aligned}\int \sin^2(x)\,dx &= \int \frac{1}{2}\left(1-\cos(2x)\right)\,dx \\ &= \frac{1}{2}\int dx - \frac{1}{2}\int \cos(2x)\,dx = \frac{x}{2} - \frac{\sin(2x)}{4}\end{aligned}$$

Thence $\displaystyle\int_{-\pi}^{+\pi} \sin^2(x)\,dx = \pi$.

Cancellation & Telescoping Sums

Example

What is $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n(n+1)}$?

$$
\begin{aligned}
\sum_{n=1}^{N} \frac{1}{n(n+1)} &= \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+1} \\
&= \left(\frac{1}{1} - \frac{1}{2}\right) + \left(\frac{1}{2} - \frac{1}{3}\right) + \left(\frac{1}{3} - \frac{1}{4}\right) + \cdots + \left(\frac{1}{N} - \frac{1}{N+1}\right) \\
&= \frac{1}{1} + \left(-\frac{1}{2} + \frac{1}{2}\right) + \left(-\frac{1}{3} + \frac{1}{3}\right) + \left(-\frac{1}{4} + \cdots + \frac{1}{N}\right) - \frac{1}{N+1} \\
&= 1 - \frac{1}{N+1}
\end{aligned}
$$

Whence

$$\sum_{n=1}^{\infty} \frac{1}{n(n+1)} = \lim_{N\to\infty} \left(1 - \frac{1}{N+1}\right) = 1$$

Bigger Telescoping Sums

Example

What is $\displaystyle\sum_{n=1}^{\infty} \frac{3}{n(n+3)}$?

$$\begin{aligned}\sum_{n=1}^{N} \frac{3}{n(n+3)} &= \sum_{n=1}^{N} \frac{1}{n} - \frac{1}{n+3} \\ &= \left(\frac{1}{1} - \frac{1}{4}\right) + \left(\frac{1}{2} - \frac{1}{5}\right) + \left(\frac{1}{3} - \frac{1}{6}\right) + \\ &\quad \left(\frac{1}{4} - \frac{1}{7}\right) + \left(\frac{1}{5} - \frac{1}{8}\right) + \left(\frac{1}{6} - \frac{1}{9}\right) + \dots \\ &= \left(\frac{1}{1} + \frac{1}{2} + \frac{1}{3}\right) - \left(\frac{1}{N+1} + \frac{1}{N+2} + \frac{1}{N+3}\right)\end{aligned}$$

Hence

$$\sum_{n=1}^{\infty} \frac{3}{n(n+3)} = \lim_{N\to\infty} \left[\frac{1}{1} + \frac{1}{2} + \frac{1}{3} - \frac{1}{N+1} - \frac{1}{N+2} - \frac{1}{N+3}\right] = \frac{11}{6}$$

Maclaurin Series

A *Maclaurin series* is a Taylor series centered at $a = 0$. Alternate techniques can be useful for finding Maclaurin expansions without searching for a formula for the nth derivative.

Example

The series for $\sec(x)$ can be found from

$$\begin{aligned}\frac{1}{\cos(x)} &= \frac{1}{1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} \pm \cdots} \\ &= 1 + \frac{1}{2}x^2 + \frac{5}{24}x^4 + \frac{61}{720}x^6 + \cdots\end{aligned}$$

To find the inverse of the cosine series, use

$$1 = \left(1 - \frac{x^2}{2!} + \frac{x^4}{4!} + O(x^6)\right) \times \left(a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 + O(x^5)\right)$$

$$1 = a_0 + a_1 x + \left(a_2 - \frac{a_0}{2}\right) x^2 + \left(a_3 - \frac{a_1}{2}\right) x^3 + \left(a_4 - \frac{a_2}{2} + \frac{a_0}{24}\right) x^4 + O(x^5)$$

$$\Rightarrow a_0 = 1 \Rightarrow a_2 = \frac{1}{2} \Rightarrow a_4 = \frac{5}{24} \Rightarrow \ldots; a_1 = 0 \Rightarrow a_3 = 0 \Rightarrow a_5 = 0, \text{ \&c}$$

Maclaurin Series, II

Example (Differentiation/Integration)

The Maclaurin series for $\ln(1+x)$ can be found as follows (subject to conditions):

$$\ln(1+x) = \int \frac{1}{1+x}\,dx$$

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - x^5 \pm \cdots$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} \pm \ldots$$

We need to investigate what conditions are needed to be able to integrate term by term; i.e., when is

$$\int \sum_{k=0}^{\infty} a_n(x)\,dx = \sum_{k=0}^{\infty} \int a_n(x)\,dx$$

permissible?

Cavalieri Sums

Faulhaber published the general formula for sums of powers in 1631 in *Academiæ Algebræ*.

Theorem

$$\sum_{k=1}^{n} k^p = \frac{1}{p+1} \sum_{j=1}^{p+1} (-1)^{\delta_{j,p}} \binom{p+1}{j} B_{(p+1-j)}\, n^j$$

where δ is the "Kronecker delta function"

$$\delta_{x,y} = \begin{cases} 1 & \text{if } x = y \\ 0 & \text{otherwise} \end{cases}$$

and B_n is nth the Bernoulli number.

Counterexamples

Studying counterexamples is important in developing a deeper understanding of concepts.

Example

- The *signum function* is $\operatorname{sgn}(x) = \begin{cases} 1 & x > 0 \\ 0 & x = 0 \\ -1 & x < 0 \end{cases}$. The signum function is not the derivative of any function. (Derivatives have the *Intermediate Value Property*.)
- Set $g(x) = |x|$ on $[-1/2, 1/2]$ and let f be its periodic extension to $\mathbb{R}$. Define

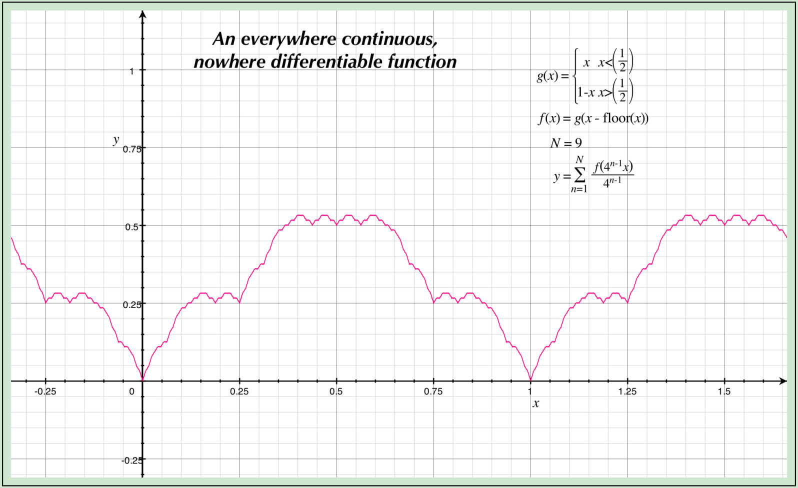
$$S(x) = \sum_{n=1}^{\infty} \frac{f(4^{n-1}x)}{4^{n-1}}$$

Then S is continuous everywhere, but differentiable nowhere.

— from Gelbaum & Olmstead's *Counterexamples in Analysis*

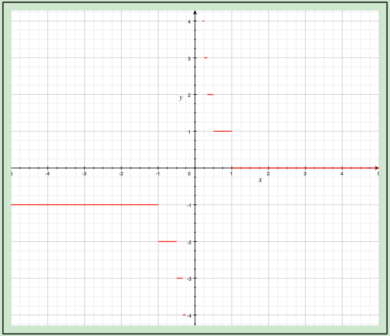
A “Nowhere Man” Function

Example

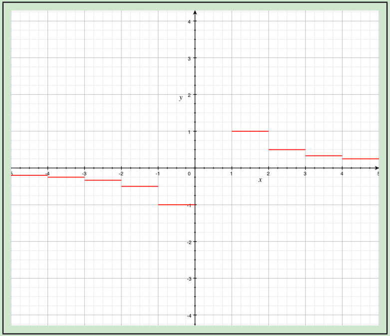


Unusual Functions

Example (Reciprocal Floors)



$$f(x) = \left\lfloor \frac{1}{x} \right\rfloor$$



$$f(x) = \frac{1}{\lfloor x \rfloor}$$

Unusual Functions, II

Example

Consider $f_n(x) = \operatorname{sgn}(x) \cdot x^n$ and it's derivatives (*if there are any*) at $x = 0$. (*Look at several cases:* $n = 2, 3, 4,$ *&c.*)

1. What is $\dfrac{d}{dx} f_n(x)$?
2. Is $\dfrac{d}{dx} f_n(x)$ continuous?
3. What is $\dfrac{d}{dx} f_n(0)$?
4. What is $\dfrac{d^n}{dx^n} f_n(x)$?
5. What is $\dfrac{d^n}{dx^n} f_n(0)$?
6. Is $\dfrac{d^n}{dx^n} f_n(x)$ continuous?

Interior, Exterior, Boundary, & Limit Points

Definition

Neighborhood A (basic) neighborhood $N(x)$ of $x \in \mathbb{R}$ is an open interval containing x.

Interior The *interior* of a set is

$$\operatorname{int}(S) = \{x | N(x) \subseteq S \text{ for some n'hood } N(x)\}.$$

Exterior The *exterior* of a set is

$$\operatorname{ext}(S) = \{x | N(x) \subseteq S^c \text{ for some n'hood } N(x)\}.$$

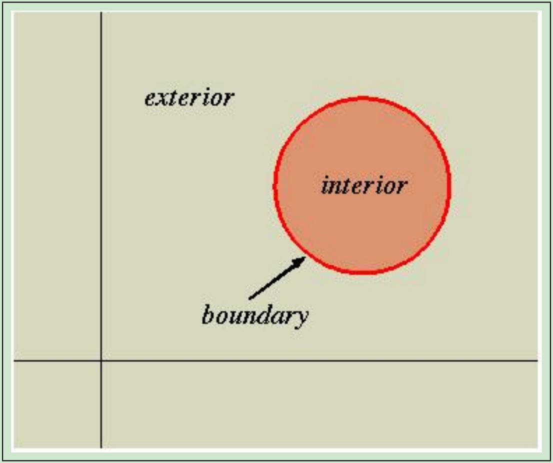
Boundary The *boundary* of a set is

$$\operatorname{bd}(S) = \mathbb{R} - (\operatorname{int}(S) \cup \operatorname{ext}(S)).$$

Limit Point The point x is a *limit point* of S iff every neighborhood $N(x)$ contains a point of S different from x; i.e., $S \cap (N(x) - \{x\}) \neq \emptyset$.

Interior/Exterior Diagram in $\mathbb{R}^2$

Example



Uniform Continuity

Definition

- A function f is *uniformly continuous on a set* S iff for any $\epsilon > 0$, there is a $\delta > 0$ such that for any $x_1, x_2 \in S$ with $|x_1 - x_2| < \delta$, we have $|f(x_1) - f(x_2)| < \epsilon$.
- A function f is *not* uniformly continuous on a set S iff there is an $\epsilon > 0$ for which any $\delta > 0$ has points $x_1, x_2 \in S$ with $|x_1 - x_2| < \delta$, but $|f(x_1) - f(x_2)| \geq \epsilon$.

Theorem

- *If f is continuous on a compact set S, then f is uniformly continuous on S.*
- *If f is uniformly continuous on (a, b), then f can be extended to be (uniformly) continuous on $[a, b]$.*
- *If f' is bounded on (a, b), then f is uniformly continuous on (a, b).*

Euler and the Sum of Reciprocal Squares

Jakob Bernoulli posed, in his 1689 *Tractatus de seriebus infinitis*, what came to be called the *Basel Problem*: Find the value of the sum

$$\sum_{k=1}^{\infty} \frac{1}{k^2}.$$

Bernoulli had shown it to be less than 2 using the inequality

$$\frac{1}{k^2} \leq \frac{1}{\frac{1}{2}\,k\,(k+1)} = \frac{2}{k} - \frac{2}{k+1}.$$

Leonhard Euler, in 1735, became the first to prove that

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{\pi^2}{6}$$

Sine as a Product and as a Series

Example

Building the sin from products

$$\frac{\sin(x)}{x} = \prod_{k=1}^{\infty}\left(1 - \frac{x^2}{k^2\pi^2}\right) = \sum_{k=1}^{\infty}(-1)^{k+1}\frac{x^{2k-2}}{(2k-1)!}$$

Euler's Second Proof

Euler was concerned about the validity of his earlier proof, so he found others.

Proof.

Lemma 1: $\dfrac{\left[\sin^{-1}(x)\right]^2}{2} = \displaystyle\int_0^x \frac{\sin^{-1}(t)}{\sqrt{1-t^2}}\,dt.$

Lemma 2: $\sin^{-1}(t) = t + \dfrac{1}{2}\cdot\dfrac{t^3}{3} + \dfrac{1\cdot 3}{2\cdot 4}\cdot\dfrac{t^5}{5} + \dfrac{1\cdot 3\cdot 5}{2\cdot 4\cdot 6}\cdot\dfrac{t^7}{7} + \cdots.$

Lemma 3: $\displaystyle\int_0^1 \frac{t^{n+2}}{\sqrt{1-t^2}}\,dt = \frac{n+1}{n+2}\int_0^1 \frac{t^n}{\sqrt{1-t^2}}\,dt$ for $n \geq 1$.

□

The Proof

Proof.

- Set $x = 1$ in Lemma 1.
- Replace the $\sin^{-1}$ term using the series in Lemma 2.
- Integrate with Lemma 3.
- These steps give $\frac{\pi^2}{8} = 1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots$; i.e., the sum of the odd squares.
- Working with the identity

$$\sum_{k=1}^{\infty} \frac{1}{k^2} = \left[1 + \frac{1}{9} + \frac{1}{25} + \frac{1}{49} + \cdots\right] + \left[\frac{1}{4} + \frac{1}{16} + \frac{1}{36} + \cdots\right]$$
$$= \frac{\pi^2}{8} + \frac{1}{4}\left[1 + \frac{1}{4} + \frac{1}{9} + \frac{1}{16} + \cdots\right]$$
$$= \frac{\pi^2}{8} + \frac{1}{4}\sum_{k=1}^{\infty} \frac{1}{k^2}$$

gives the result. $\square$

Interlude: Euler and Polynomial Roots

Theorem

Suppose the monic nth degree polynomial

$$p(x) = x^n - Ax^{n-1} + Bx^{n-2} - Cx^{n-3} + \cdots \pm N$$

factors as $p(x) = (x - r_1)(x - r_2) \cdots (x - r_n)$. Then

$$\sum_{k=1}^{n} r_k = A$$

$$\sum_{k=1}^{n} r_k^2 = A \sum_{k=1}^{n} r_k - 2B$$

$$\sum_{k=1}^{n} r_k^3 = A \sum_{k=1}^{n} r_k^2 - B \sum_{k=1}^{n} r_k + 3C$$

$$\sum_{k=1}^{n} r_k^4 = A \sum_{k=1}^{n} r_k^3 - B \sum_{k=1}^{n} r_k^2 + C \sum_{k=1}^{n} r_k - 4D$$

$$\vdots$$

Sequences & Series of Functions

Definition (Pointwise Convergence)

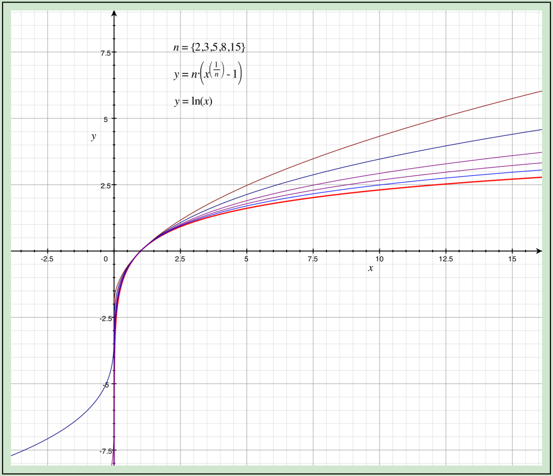
- A sequence of functions $\{f_n\}$ converges to $f(x)$ at a point $x \in \text{dom}(f)$ iff for any $\epsilon > 0$, there is an $N = N(x, \epsilon) \in \mathbb{N}$ such that $n > N$ implies $|f_n(x) - f(x)| < \epsilon$.
- A series of functions $\sum_{k=0}^{\infty} f_k$ converges to $f(x)$ at a point $x \in \text{dom}(f)$ iff for any $\epsilon > 0$, there is an $N = N(x, \epsilon) \in \mathbb{N}$ such that $n > N$ implies $|\sum_{k=0}^{n} f_k(x) - f(x)| < \epsilon$.

Definition (Uniform Convergence)

- A sequence (series) of functions $\{S_n\}$ converges uniformly to $S(x)$ for every $x \in \text{dom}(S)$ iff for any $\epsilon > 0$, there is an $N = N(\epsilon) \in \mathbb{N}$ such that $n > N$ implies $|S_n(x) - f(x)| < \epsilon$.

A Sequence of Functions

Example



Uniform Convergence and Integration

Theorem

If f_n is integrable on $[a, b]$ for all n and $f_n \to f$ uniformly, then f is integrable and

$$\int_a^b \lim_{n\to\infty} f_n(x)dx = \lim_{n\to\infty} \int_a^b f_n(x)dx.$$

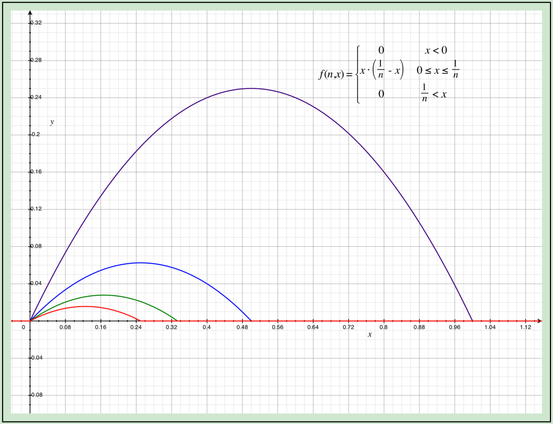
Theorem

If f_n is integrable on $[a, b]$ for all n and $\sum f_n$ converges uniformly, then the sum is integrable and

$$\int_a^b \sum_{n=0}^{\infty} f_n(x)dx = \sum_{n=0}^{\infty} \int_a^b f_n(x)dx.$$

A Good Sequence of Functions

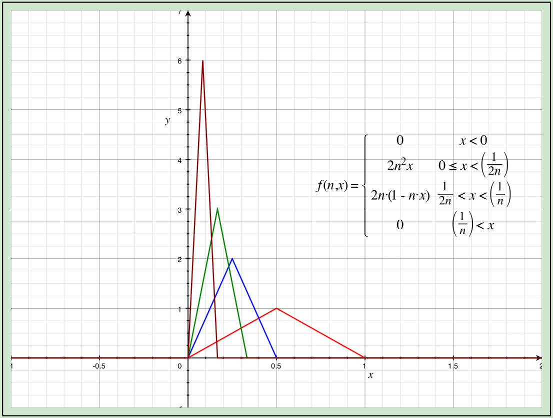
Example



- Find $\int f_n$, $\lim_n f_n$, $\lim_n \int f_n$, and $\int \lim_n f_n$.

A Bad Sequence of Functions

Example



- Find $\int f_n$, $\lim_n f_n$, $\lim_n \int f_n$, and $\int \lim_n f_n$.

Uniform Convergence and Differentiation

Differentiation does not behave as well as integration.

Example

Let $f_n(x) = \frac{1}{n}\sin\left(n^2 x\right)$. Show that

- f_n converges uniformly on $\mathbb{R}$
- f_n' doesn't even converge pointwise anywhere.

Theorem

Suppose $f_n : [a,b] \to \mathbb{R}$ is differentiable for all n and $f_n(x_0)$ converges for some point $x_0 \in [a,b]$. If f_n' converges uniformly on $[a,b]$, then f_n converges uniformly on $[a,b]$ and

$$\frac{d}{dx}\lim_{n\to\infty} f_n(x) = \lim_{n\to\infty}\frac{d}{dx} f_n(x).$$

The Best Uniform Convergence Test

The Weierstrass M–test provides a very useful method for testing uniform convergence.

Theorem (The Weierstrass M–Test)

Let f_n be a sequence of functions. If there is a sequence of constants M_n such that

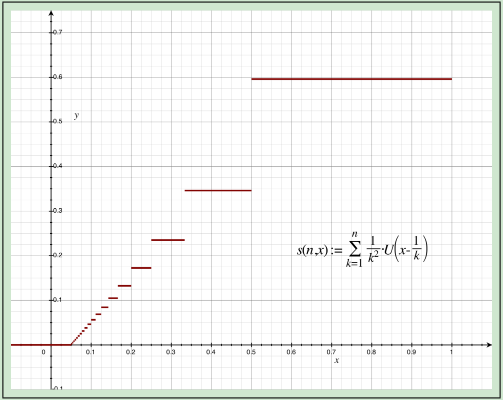
- $|f_n(x)| \leq M_n$ *for all* $n \in \mathbb{N}$ *and*
- $\displaystyle\sum_{n=1}^{\infty} M_n$ *converges,*

then $\displaystyle\sum_{n=1}^{\infty} f_n$ *converges uniformly (and absolutely).*

1. Does $\sum_n f_n(x)$ converge where $f_n(x) = \dfrac{1}{n^2}\sin\left(n^2 x\right)$?
2. Why doesn't the test work for f_n'?

A Series of Steps

Example



$$S(x) = \sum_{k=1}^{\infty} \frac{1}{k^2} \cdot U\left(x - \frac{1}{k}\right)$$

The Rationals Are a Small Set

Theorem

The rationals are countable.

Proof.

Let $\mathbb{Q}$ *represent the set of rational numbers. The array below shows a method of enumerating all the rationals.*

$$\begin{array}{ccccc} 1/1_{(1)} & 2/1_{(2)} & 3/1_{(4)} & 4/1_{(7)} & \cdots \\ 1/2_{(3)} & 2/2_{(5)} & 3/2_{(8)} & 4/2_{(12)} & \cdots \\ 1/3_{(6)} & 2/3_{(9)} & 3/3_{(13)} & 4/3_{(18)} & \cdots \\ 1/4_{(10)} & 2/4_{(14)} & 3/4_{(19)} & 4/4_{(25)} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{array}$$

Since each rational is counted, we have $|\mathbb{Q}| \le |\mathbb{N}|$ *where we use* $|\cdot|$ *to indicate cardinality (or size). But we know that* $\mathbb{N} \subseteq \mathbb{Q}$, *so that* $|\mathbb{N}| \le |\mathbb{Q}|$. *Hence* $|\mathbb{Q}| = |\mathbb{N}|$. □

Open Covers

Definition

An *open cover* of a set A is a collection of open sets $\{O_n \,|\, n \in \mathcal{N}\}$ such that

$$A \subseteq \bigcup_{n \in \mathcal{N}} O_n.$$

Example

- The collection $\{(0,2),(3,4)\}$ is an open cover of $A = [0.5, 1.5] \cup \{3.25, 3.5, 3.75\}$.
- The collection $\{(n - 1/4, n + 1/4) \,|\, n \in \mathbb{N}\}$ is an open cover of $\mathbb{N}$.
- $\{\mathbb{R}\}$ is an open cover of $\mathbb{R}$.

Covering $\mathbb{Q}$

Theorem

The rationals have an open cover of arbitrarily small total length.

Proof.

Let $\epsilon > 0$. *List the rationals in order* $\mathbb{Q} = \{r_1, r_2, r_3, \ldots\}$ *as given by the "countability matrix" defined earlier. For each rational* r_k, *define the open interval* $I_k = (r_k - \epsilon/2^{k+1}, r_k + \epsilon/2^{k+1})$. *Then*

- *the collection* $\mathcal{C} = \{I_k \mid k \in \mathbb{N}\}$ *forms an open cover of* $\mathbb{Q}$,
- *the length of each* I_k *is* $m(I_k) = \epsilon/2^k$.

The total length of the intervals in $\mathcal{C}$ *is*

$$m(\mathcal{C}) = \sum_{k=1}^{\infty} m(I_k) = \sum_{k=1}^{\infty} \epsilon/2^k = \epsilon \sum_{k=1}^{\infty} \frac{1}{2^k} = \epsilon$$

□

Measure Zero

Definition

A set $S \subset \mathbb{R}$ has *measure zero*, written as $m(S) = 0$, if and only if for any $\epsilon > 0$ there is an open cover $\mathcal{C} = \{O_k \mid k \in \mathcal{N}\}$ of S such that $\sum_{k \in \mathcal{N}} m(O_k) < \epsilon$.

Example

1. The rationals have measure zero.
2. Any finite set has measure zero.
3. Every interval $[a, b]$ is not measure zero (when $a < b$).

- Assume the measure of $[0, 1]$ is 1. The rationals contained in $[0, 1]$ have measure zero. What do you conjecture the measure of the irrationals in $[0, 1]$ is?

A Brief Introduction to Lebesgue Measure

The Riemann integral cannot handle functions like Dirichlet's everywhere discontinuous *characteristic function* of the rationals; i.e., $\chi(x) = \{1 \text{ if } x \in \mathbb{Q}, 0 \text{ otherwise}\}$. Lebesgue introduced a *measure* based on the length of intervals containing the set. There are sets that cannot be measured (but that is beyond our scope). Lebesgue's measure has the following properties:

Theorem

Let S, S_n, and T all be measurable, then:

1. $\mu(S) \geq 0$.
2. *If* $S \subseteq T$, *then* $\mu(S) \leq \mu(T)$.
3. *If* $S \cap T = \emptyset$, *then* $\mu(S \cup T) = \mu(S) + \mu(T)$.
4. $\mu(S \cup T) \leq \mu(S) + \mu(T)$.
5. $\mu\left(\bigcup_{n=1}^{\infty} S_n\right) \leq \sum_{n=1}^{\infty} \mu(S_n)$.

Building Lebesgue Measure

The basic idea is to start with intervals and use open covers to build to more complex sets. (We won't go into real generality.)

Definition

- Define $\mu(I) = b - a$ for the open interval $I = (a, b)$ where $a \leq b$.
- For a set E, define $\mu^*(E) = \inf_{\mathcal{O}} \sum \mu(I_n)$ where $\mathcal{O} = \{I_n\}$ forms an open cover of E.
- Define $\mu(E) = \mu^*(E)$ and call E *measurable* iff for each set A, we have
$$\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^c).$$

1. Show that $\mu([a, b]) = b - a$.
2. Show that $\mu(\{x_1, x_2, \ldots, x_n\}) = 0$ for any finite set..

Theorem (*TFAE*)

1. *E is measurable.*
2. *Given $\epsilon > 0$, there is an open set G such that $\mu^*(G - E) < \epsilon$.*
3. *Given $\epsilon > 0$, there is an closed set F such that $\mu^*(E - F) < \epsilon$.*

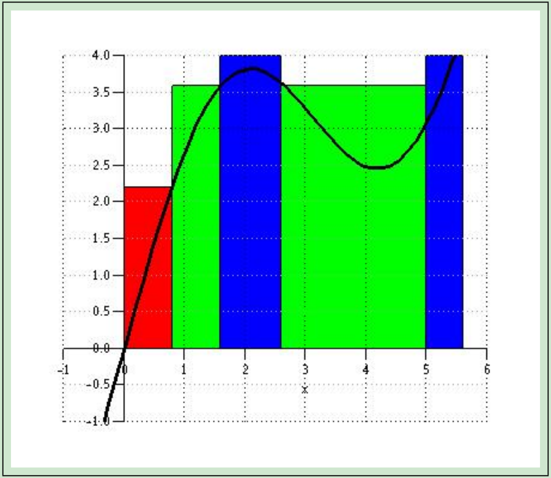
Building the Lebesgue Integral

Definition

- The *characteristic function* of a set E is
$$\chi_E(x) = \{1 \text{ if } x \in E,\ \ 0 \text{ otherwise}\}.$$
- A *simple function* is a function of the form
$$\phi(x) = \sum_{k=1}^{n} a_i \cdot \chi_{E_i}(x)$$
where each E_i is measurable and n is finite.
- If ϕ and $E = \cup E_k$ are bounded, define
$$\int_E \phi = \sum_{k=1}^{n} a_i \cdot \mu(E_i)$$
- If f is measurable and bounded on a bounded set E, define
$$\int_E f = \inf_{\phi} \int_E \phi$$
for all simple functions $\phi \geq f$.

Lebesgue Integral Example

Example



Special Functions — The Gamma Function

Recall the definition of the Gamma function.

Definition

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt$$

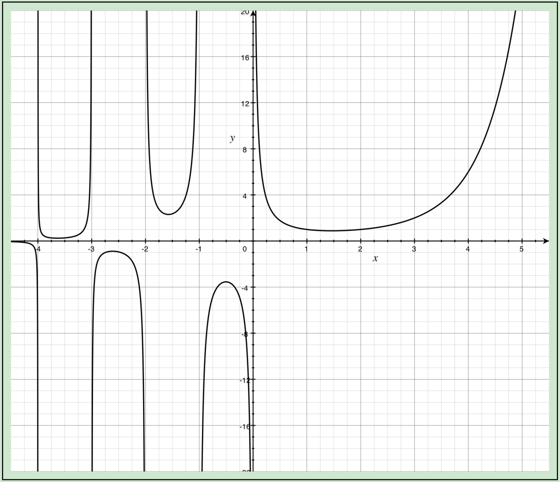
This special function that extends the factorial has many interesting properties.

Theorem

- $\Gamma(1/2) = \sqrt{\pi}$ *(u-substitution)*
- $\Gamma(x+1) = x \cdot \Gamma(x)$ *for* $x > 0$ *(integrate by parts)*
- $\Gamma(n+1) = n!$ *(recursion)*
- $\Gamma(x+1) \approx (x/e)^x \sqrt{2\pi x}$ *which implies* $n! \approx (n/e)^n \sqrt{2\pi n}$ *(Stirling's formula)*

The Gamma Function

Example



A plot of $\Gamma(x)$ in $[-4.5, 5] \times [-20, 20]$.

Fourier Series

Jean Baptiste Joseph Fourier developed trigonometric series as representations/approximations for, he claimed, any periodic function in his 1822 book *Théorie analytique de la chaleur* (*Analytical Theory of Heat*). He was surprisingly close to being right.

Definition

The *Fourier series* of a 2π-periodic function $f(x)$ is given by[2]

$$\hat{f}(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

Some texts replace a_0 with $a_0/2$ for convenience.
(In 1824, he postulated warming of the atmosphere by gases which is now called the *greenhouse effect.*)

[2]Most texts use a_n with $\cos$ and b_n with $\sin$.

The Fourier Coefficients

In order to calculate the Fourier coefficients, we need several facts that we have already shown to be true (cf. pg. 21).

Theorem

- $\int_{-\pi}^{+\pi} \sin(nx)\cos(kx)\,dx = 0$ *for all* n *and* k.
- $\int_{-\pi}^{+\pi} \sin(nx)\sin(kx)\,dx = 0$ *for* $n \neq k$.
- $\int_{-\pi}^{+\pi} \cos(nx)\cos(kx)\,dx = 0$ *for* $n \neq k$.
- $\int_{-\pi}^{+\pi} \sin^2(nx)\,dx = \begin{cases} \pi & n \neq 0 \\ 0 & n = 0 \end{cases}$
- $\int_{-\pi}^{+\pi} \cos^2(nx)\,dx = \begin{cases} \pi & n \neq 0 \\ 2\pi & n = 0 \end{cases}$

Fourier's Computation, I

Fourier's calculations run roughly as follows:
Multiply the series by $\cos(nx)$

$$f(x)\cos(nx) = a_0\cos(nx) + \sum_{k=1}^{\infty} a_k\cos(kx)\cos(nx) + b_k\sin(kx)\cos(nx)$$

Integrate from $-\pi$ to $+\pi$

$$\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx = \int_{-\pi}^{+\pi} a_0\cos(nx)\,dx + \int_{-\pi}^{+\pi}\sum_{k=1}^{\infty} a_k\cos(kx)\cos(nx) + b_k\sin(kx)\cos(nx)dx$$

Interchange operations (!) (*What conditions are necessary here?*)

$$\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx = \int_{-\pi}^{+\pi} a_0\cos(nx)\,dx + \sum_{k=1}^{\infty}\left[\int_{-\pi}^{+\pi} a_k\cos(kx)\cos(nx)\,dx + \int_{-\pi}^{+\pi} b_k\sin(kx)\cos(nx)\,dx\right]$$

Fourier's Computation, II

Now we apply the previous facts to see all the terms disappear but for the one with $k = n$

$$\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx = \int_{-\pi}^{+\pi} a_0 \cos(nx)\,dx + \int_{-\pi}^{+\pi} a_n \cos(nx)\cos(nx)\,dx + \int_{-\pi}^{+\pi} b_k \sin(nx)\cos(nx)\,dx$$

If $n > 0$, then

$$\int_{-\pi}^{+\pi} f(x)\cos(nx)\,dx = \int_{-\pi}^{+\pi} a_n \cos^2(nx)\,dx = a_n\pi$$

If $n = 0$, then

$$\int_{-\pi}^{+\pi} f(x)\,dx = \int_{-\pi}^{+\pi} a_0\,dx = a_0\,2\pi$$

Solve for a_n and a_0, respectively. Do the same for b_n.

Fourier's Computation, III

Based on Fourier's calculations, we arrive at

Definition

The *Fourier series* of a 2π-periodic function $f(x)$ is given by

$$\hat{f}(x) = a_0 + \sum_{k=1}^{\infty} a_k \cos(kx) + b_k \sin(kx)$$

where

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{+\pi} f(x)\, dx$$

$$a_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \cos(nx)\, dx, \quad n > 0$$

$$b_n = \frac{1}{\pi} \int_{-\pi}^{+\pi} f(x) \sin(nx)\, dx, \quad n > 0$$

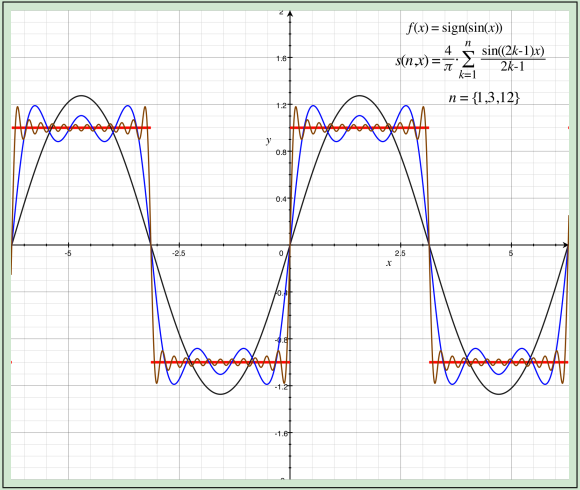
Series Examples

$f(x) = x$	$\hat{f}(x) = 2\sum_{n=1}^{\infty}(-1)^{n+1}\frac{\sin(nx)}{n}$
$f(x) = \|x\|$	$\hat{f}(x) = \frac{\pi}{2} - \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\cos((2n-1)x)}{(2n-1)^2}$
$f(x) = \begin{cases} +1 & 0 < x < \pi \\ -1 & -\pi < x < 0 \end{cases}$	$\hat{f}(x) = \frac{4}{\pi}\sum_{n=1}^{\infty}\frac{\sin((2n-1)x)}{(2n-1)}$
$f(x) = x^2$	$\hat{f}(x) = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty}(-1)^n\frac{\cos(nx)}{n^2}$
$f(x) = \sin^2(x)$	$\hat{f}(x) = \frac{1}{2} - \frac{1}{2}\cos(2x)$

Table: Several Fourier Series

A Fourier Series

Example



A plot of $f(x)$ and 3 Fourier approximants in $[-\pi, +\pi] \times [-2, 2]$.