USING MAPLE IN ADVANCED CALCULUS AND MODERN ALGEBRA

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Abstract

We present a discussion and demonstration of enhancing upper-level courses such as Modern Algebra and Advanced Calculus with Maple. Worksheets with embedded components that allow students to focus on concepts, rather than syntax, are highlighted.

Introduction

Maple can be used in many ways to create *MicroWorlds* for students to use to explore concepts. A MicroWorld is the same concept as a "sandbox" in software programming. The MicroWorld defines parameters and operations that a student can perform in a structured environment. There are significant advantages to this approach.

1. Possible operations are restricted to the desired task and can be made context sensitive.
2. Learning advanced syntax is eliminated and can actually be hidden.

MicroWorlds can be created using a number of different Maple constructs. A primitive form is to provide students with a Maple worksheet with all output deleted. A better method is to give students a Maple document that contains automatically executed initialization code. A very useful technique is to build Maplets. A Maplet is a Java front-end to Maple that can present windows containing graphs, buttons, sliders, input/output boxes, and several other interface elements. See "Maplets for Calculus" for an extensive set of award-winning Maplets for elementary calculus students created by Doug Meade of USC and Phil Yasskin of Texas A&M.

The technique we will focus on in this session is using embedded components in a Maple document. Embedded components can be buttons, combo boxes, check boxes, radio buttons, labels, text areas, sliders, various gauges, graphs, math/text displays, or code blocks. Most components have associated actions that can be executed via some trigger. For example, changing the position of a slider changes the value of its associated variable. When the specific trigger action occurs, a set of Maple statements can be executed. The Maple code can perform calculations and update any active embedded components in the document. Programmatic changes to components are handled with the DocumentTools package. We will center our attention on two junior level courses Modern Algebra and Advanced Calculus for our examples.

Modern Algebra

Many Modern Algebra courses begin by extending familiar structures such as the integers. An early topic is the Euclidean algorithm. Students are familiar with the mechanics of the division algorithm, but are likely to have never formally considered it. We wish them to make connections between these algorithms, linear combinations, and the greatest common divisor. Numerical experiments can help students build intuition.

The Euclidean Algorithm

We present two views: the tabular form of the Euclidean Algorithm and a list of the smallest coefficients for linear combinations of the two numbers that gives the GCD. In Figure 1 we see a "code block" icon

at the upper left followed by the label `Initialize Euclidean Algorithm`. A code block is a Maple 12 construct that contains a block of Maple code which is executed whenever the icon is clicked. Here it is used to define the procedures, set up the arrays, and assign the variables needed by the associated components.
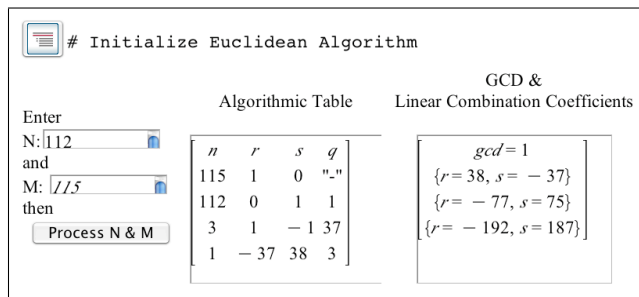
```
# Initialize Euclidean Algorithm
```

Enter
N: 112
and
M: 115
then
[Process N & M]

Algorithmic Table

$$\begin{bmatrix} n & r & s & q \\ 115 & 1 & 0 & \text{"-"} \\ 112 & 0 & 1 & 1 \\ 3 & 1 & -1 & 37 \\ 1 & -37 & 38 & 3 \end{bmatrix}$$

GCD &
Linear Combination Coefficients

$$\begin{bmatrix} gcd = 1 \\ \{r = 38, \, s = -37\} \\ \{r = -77, \, s = 75\} \\ \{r = -192, \, s = 187\} \end{bmatrix}$$

Figure 1: Euclidean Algorithm Maple Components

Cayley Tables for $\mathbb{Z}_n$

Operation tables provide a natural lede to developing the definition of a group. We start students' investigations with $\mathbb{Z}_n$, hoping to lay a basis for more abstract structures and permutations groups soon. Having students produce Cayley tables for $\mathbb{Z}_n$ is time consuming and prone to error. Maple is quicker and more accurate. A bonus is that we can instantly produce different size tables during a class session.
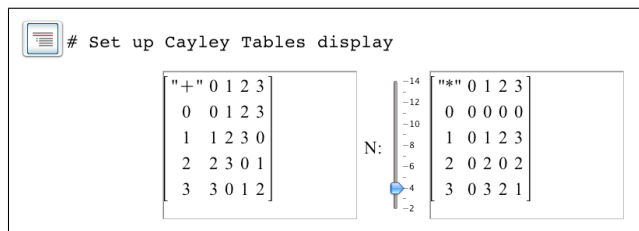
```
# Set up Cayley Tables display
```

$$\begin{bmatrix} \text{"+"} & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 2 & 3 & 0 \\ 2 & 2 & 3 & 0 & 1 \\ 3 & 3 & 0 & 1 & 2 \end{bmatrix}$$

N:

$$\begin{bmatrix} \text{"*"} & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 0 & 2 & 0 & 2 \\ 3 & 0 & 3 & 2 & 1 \end{bmatrix}$$

Figure 2: Cayley Tables Maple Components

Galois Fields

At the end of an introductory course, there is hopefully time for advanced topics to whet appetites for further study. One of my favorite topics is a low-level introduction to Galois fields. This subject gives me an opportunity to show mathematics that is currently being used daily as Galois fields are at the juncture of algebra and cryptography. The *Advanced Encryption System* (AES) of the US is based on the Galois field of $GF(2^8)$ elements. For the algebraists in the audience, AES is based on $\mathbb{Z}_2[x]/\langle x^8 + x^4 + x^3 + x + 1\rangle$.

Galois Field Operation Tables

While the full operation tables are too large to be useful for students, smaller segments show interesting properties. Figure 3 shows a variable size segment of $GF(2^8)$.
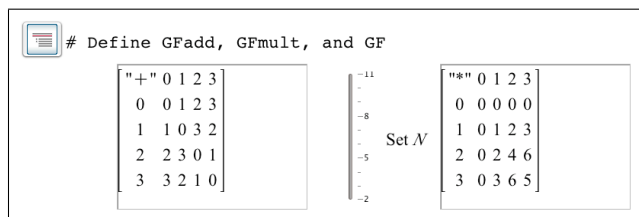
```
# Define GFadd, GFmult, and GF
```

$$\begin{bmatrix} \text{"+"} & 0 & 1 & 2 & 3 \\ 0 & 0 & 1 & 2 & 3 \\ 1 & 1 & 0 & 3 & 2 \\ 2 & 2 & 3 & 0 & 1 \\ 3 & 3 & 2 & 1 & 0 \end{bmatrix}$$

Set N

$$\begin{bmatrix} \text{"*"} & 0 & 1 & 2 & 3 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 2 & 3 \\ 2 & 0 & 2 & 4 & 6 \\ 3 & 0 & 3 & 6 & 5 \end{bmatrix}$$

Figure 3: Galois Field Maple Components

$GF(2^8)$ **Calculator**

Since our tables are too small to be really useful for calculations, we embed a small calculator that shows both the representation of an integer in $GF(2^8)$ and the results of the three standard arithmetic operations. The calculator lets students experiment with different values, representations, and arithmetic operations in the Galois field. The tie to binary is then easy for students to uncover.

Figure 4: $GF(256)$ Calculator Maple Components

Advanced Calculus

Typically, students find the rigour of ϵ-δ arguments to be extremely mysterious in an Advanced Calculus course. Graphically relating the logic to convergence often helps students to build intuition.

Sequential Limits

In Maple, define a sequence a_n as a function $a(n)$. We'll use `a(n):=sin(n^2)/n` to illustrate the components.

Graphical Tool

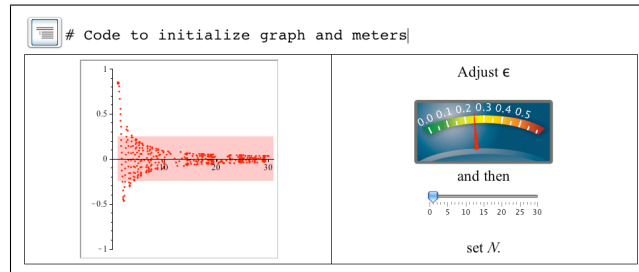A dynamic plot is much more effective at illustrating the arguments than a drawing. (See Figure 5.) Adjust

Figure 5: Sequence Graphing Maple Components

ϵ by moving the meter's needle. Then adjust N with the slider. The "ϵ-strip" changes from red to blue when an appropriate N is chosen.

A Proof in a Maple Document

Now we prove that $\left|\sin(n^2)/n\right| \to 0$ as $n \to \infty$. (*The following argument is a LATEX export from Maple.*)

Proof. Let $\epsilon > 0$. Consider the inequality

$|a(n)| < \epsilon$

$$\left|\frac{\sin\left(n^2\right)}{n}\right| < \epsilon$$

Use the fact that $\sin$ is bounded by 1 so

$subs\left(\sin\left(n^2\right) = 1, \%\right)$

$$\left|n^{-1}\right| < \epsilon$$

$simplify(abs(1/n) < \epsilon)$ assuming n :: posint

$$n^{-1} < \epsilon$$

Choosing N to be greater than $1/\epsilon$ assures that $|a_n - 0| < \epsilon$, and the result is seen to hold.

Function Limits

We will consider the function $f(x) = x^3 - x$ to demonstrate investigating the limit as x approaches -1. While this function is simple, the calculations can get in the way for student learning limits and ϵ-δ proofs.

Graphical Tool

In the same fashion as with sequential limits, a dynamic graphical display helps our students to develop intuition into the process. This intuition then, in turn, helps them to understand and to write ϵ-δ proofs. Figure 6 shows that we use a vertical slider for ϵ this time and a text box beneath the δ slider to indicate
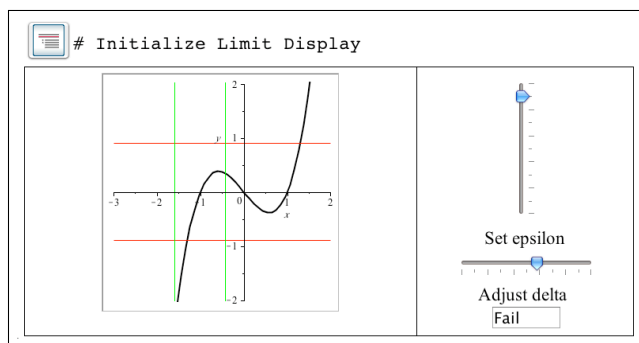

Figure 6: Function Limit Graphing Maple Components

an appropriate choice by changing from `Fail` to `Success`.

A Proof in a Maple Document

We begin by proposing the value 0 for the limit. (*The following argument is a LaTeX export from Maple*.)

$L := 0 :$
Consider
$|f(x) - L|$

$$|x^3 - x|$$

Factor to see
$factor\left(|x^3 - x|\right)$

$$|x(x-1)(x+1)|$$

If $\delta < 1$, then $|x - 1| < \delta$ implies that $0 < x < 2$, that $-1 < x - 1 < 1$, and that $1 < x + 1 < 3$. Hence $0 < x(x-1)(x+1) < 6\delta$. Thus $|f(x) - L| < 6\,\delta$. So choose $\delta < \min(1, 1/6\,\epsilon)$ and the limit $L = 0$ is verified.

Riemann-Stieltjes Integrals

Another of my favorite advanced topics to end an introductory advanced calculus course with is Riemann-Stieltjes integration. This integral introduces students to 20th century analysis. Students begin to see analysis as developing, not a subject that began and ended with Newton and Leibniz. We are investigating measures that have derivatives.

Both Courses

Often, Mathematical Induction appears in both Modern Algebra and Advanced Calculus. Most instructors agree the duplication is good for students and enjoy the topic so that "doing it twice" is a bonus.
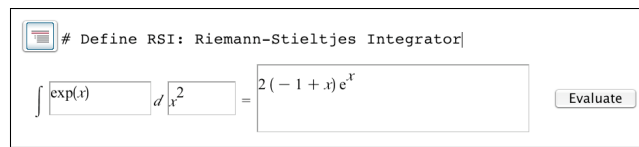
Figure 7: Riemann-Stieltjes Integrator Components

Mathematical Induction

Many student see induction proofs for the first time in either Modern Algebra or Advanced Calculus. Even though the draft is long gone, they are still initially afraid of the term *induction*. Some get lost in arithmetic errors, instead of concentrating on the method. Others try to be algorithmic, rather than thinking. Using Maple let's us off-load the computation and focus on the concept. The arithmetic sum formulas are favourite exercises for most instructors. Another benefit is that in Maple we can define a proposition of n as an equation.

A Sample Induction Homework Problem

Define $P(n)$ to be the proposition
```
> P(n) := Sum(j^3, j=1..n) = 1/4 *n^2*(n+1)^2:
```
The Basis Step. First, demonstrate that an initial case is true
```
> P(1)
```

$$\sum_{j=1}^{1} j^3 = 1$$

The Induction Step. Now we assume that $P(k)$ holds and determine that then $P(k+1)$ must also be true. What is $P(k+1)$?
```
> P(k+1)
```

$$\sum_{j=1}^{k+1} j^3 = \frac{1}{4} \cdot (k+1)^2 \cdot (k+2)^2$$

Add $(k+1)^3$ to both sides of $P(k)$.
```
> P(k) + (k+1)^3
```

$$\left(\sum_{j=1}^{k+1} j^3 \right) + (k+1)^3 = \frac{1}{4} \cdot (k+1)^2 \cdot (k+2)^2 + (k+1)^3$$

and factor the right side
```
> factor(rhs(%))
```

$$\frac{1}{4} \cdot (k+1)^2 \cdot (k+2)^2$$

Comparing the expressions shows that the induction holds.

Conclusion

Embedded components can be used to build MicroWorlds for student investigations. Actions attached to embedded objects remove the need for students to learn complex syntaxes and allow them to focus on the concepts while also providing a richer environment for study and experimentation. Students can build a stronger understanding and develop deeper intuition using this facility of Maple.

For an online video tutorial on embedded components from Maplesoft, visit:
http://www.maplesoft.com/support/training/videos.aspx

Maple 11 and 12 worksheets for this document are available at:
www.mathsci.appstate.edu/~wmcb/ICTCM/ICTCM20/