Semantics of Advanced Data Types

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Course Outline

Lecture 1: Syntax and semantics of ADTs and nested types ✓

Lecture 2: Syntax and semantics of GADTs

Lecture 3: Parametricity for ADTs and nested types

Lecture 4: Parametricity for GADTs

Lecture 2:
Syntax and Semantics of GADTs

ADTs $\xhookrightarrow[\text{generalized by}]{\text{syntactically}}$ nested types $\xhookrightarrow[\text{generalized by}]{\text{syntactically}}$ GADTs

✓ ✓

What is a GADT?

- The shape of a GADT structure can depend on the data it contains.
- GADT data constructors can have both input types *and return types* involving instances of the data type being defined other than the one being defined.
- Fancier constructor types mean that GADTs can encode more sophisticated correctness properties.
- Sequences

```
data Seq : Set → Set where
  const : ∀{A : Set} → A → Seq A
  spair  : ∀{A B : Set} → Seq A × Seq B → Seq (A × B)
```

  Note that spair only constructs sequences of pair types.

- Polynomial expressions with variables of type A and coefficients of type B

```
data Expr : Set → Set → Set where
  var      : ∀{A B : Set} → A → Expr A B
  iconst   : ∀{A : Set} → Int → Expr A Int
  fconst   : ∀{A : Set} → Float → Expr A Float
  prod     : ∀{A B : Set} → Expr A B → Expr A B → Expr A B
  iscmult : ∀{A B : Set} → Expr A B → Int → Expr A B
  fscmult : ∀{A B : Set} → Expr A B → Float → Expr A Float
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\text{data Seq} : \text{Set} \to \text{Set where} \\
\quad \text{const} : \forall\{A : \text{Set}\} \to A \to \text{Seq } A \\
\quad \text{spair} \;\; : \forall\{A\, B : \text{Set}\} \to \text{Seq } A \times \text{Seq } B \to \text{Seq } (A \times B)
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GADTs Are Not Functorial

- GADTs were functorial, they'd have shape-preserving, data-changing map functions.
- Consider $\text{map}_{\text{Seq}} : (A \to B) \to \text{Seq}\, A \to \text{Seq}\, B$
- The clause of map for const should have

$$\text{map}_{\text{Seq}}\ f\ (\text{const}\ x) = \text{const}\ (f\ x)$$

- What should the clause of map for spair be? If $f : C \times D \to E$ then

$$\text{map}_{\text{Seq}}\ f\ (\text{spair}\ s_1\ s_2) = \text{spair}\ ?\ ?$$

- What if $E \neq U \times V$?
- What if $E = U \times V$ but $f \neq (g : C \to U) \times (h : D \to V)$?
- Similarly, we can't construct the clause of map_{Expr} for iconst, fconst, or fscmult.
- GADTs do not support map functions because they are not data types in the usual container-y sense.
- Question: How do we give initial algebra semantics to GADTs?

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Recovering Functoriality

- There are two ways to recover functoriality. Both can be described in terms of left Kan extensions.
- The left Kan extension of $F : \mathcal{C} \to \mathcal{D}$ along $K : \mathcal{C} \to \mathcal{E}$ — denoted $Lan_K F$ — gives the "best functorial approximation" to F that factors through K.
- Intuitively, this means that $Lan_K F$ is the smallest functor that both extends the image of K to $\mathcal{D}$ and is such that the extension $Lan_K F \circ K$ agrees with F on $\mathcal{C}$, in the sense that there is a natural transformation η from F to $Lan_K F \circ K$.
- "Smallest" means that, for any other such extension G, there is a unique natural transformation δ from $Lan_K F$ to G such that the two natural transformations η and γ out of F are related nicely.

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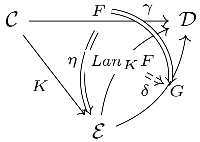

Left Kan Extensions

- If $F : \mathcal{C} \to \mathcal{D}$ and $K : \mathcal{C} \to \mathcal{E}$ are functors, then the *left Kan extension of F along K* is a functor $Lan_K\, F : \mathcal{E} \to \mathcal{D}$ together with a natural transformation $\eta : F \to Lan_K\, F \circ K$ such that, for every functor $G : \mathcal{E} \to \mathcal{D}$ and natural transformation $\gamma : F \to G \circ K$, there exists a unique natural transformation $\delta : Lan_K\, F \to G$ such that $(\delta K) \circ \eta = \gamma$.
- There is an isomorphism of natural transformations

$$F \to G \circ K \;\cong\; Lan_K F \to G$$

- If we add to our type system a type constructor Lan that is the syntactic reflection of the categorical Lan, then we can use (the syntactic reflection of) the above isomorphism to rewrite the syntax of our GADTs.
- This gives a "best approximation" *functorial completion* of GADT syntax that lets us rewrite GADT data constructor types in the same form as the types of data constructors for nested types.
- Functional completion lets us model GADTs as fixpoints of higher-order functors.

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- This gives a "best approximation" *functorial completion* of GADT syntax that lets us rewrite GADT data constructor types in the same form as the types of data constructors for nested types.
- Functional completion lets us model GADTs as fixpoints of higher-order functors.

Rewriting GADT Syntax (I)

- We can rewrite Seq as follows:

$$
\begin{array}{l}
\mathsf{data\ Seq} : \mathsf{Set} \to \mathsf{Set\ where} \\
\quad \mathsf{const} : \forall\{\mathsf{A} : \mathsf{Set}\} \to \mathsf{A} \to \mathsf{Seq\,A} \\
\quad \mathsf{spair} \ : \forall\{\mathsf{A\,B} : \mathsf{Set}\} \to \underbrace{Seq\,A \times Seq\,B}_{\mathsf{F\,A\,B}} \to \underbrace{Seq}_{\mathsf{G}}\,(\underbrace{A \times B}_{\mathsf{K\,A\,B}}) \\
\quad \mathsf{spair} \ : \forall\{\mathsf{A} : \mathsf{Set}\} \to (\mathsf{Lan}_{\lambda \mathsf{A\,B}.\, \mathsf{A} \times \mathsf{B}}\, \lambda \mathsf{A\,B}.\, \mathsf{Seq\,A} \times \mathsf{Seq\,B})\,\mathsf{A} \to \mathsf{Seq\,A}
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- Then Seq can be interpreted as μH for the higher-order functor

$$H\,F\,X = X + (Lan_{\lambda XY.\,X \times Y}\, \lambda XY.\, F X \times F Y)\, X$$

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\quad \text{spair} \;\; : \forall\{\text{A} : \text{Set}\} \to (\text{Lan}_{\lambda \text{A B. A} \times \text{B}}\, \lambda \text{A B. Seq A} \times \text{Seq B})\, \text{A} \to \text{Seq A}
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Rewriting GADT Syntax (II)

- We can rewrite Expr as follows:

  data Expr : Set → Set → Set where
  var : ∀{A B : Set} → A → Expr A B
  iconst : ∀{A : Set} → Int → Expr A Int
  iconst : ∀{A B : Set} → $(\mathsf{Lan}_{\lambda A\,B.A \times \mathsf{Int}}\ \lambda A\,B.\ \mathsf{Int})\ A\ B$ → Expr A B
  fconst : ∀{A : Set} → Float → Expr A Float
  fconst : ∀{A B : Set} → $(\mathsf{Lan}_{\lambda A\,B.A \times \mathsf{Float}}\ \lambda A\,B.\ \mathsf{Float})\ A\ B$ → Expr A B
  prod : ∀{A B : Set} → Expr A B → Expr A B → Expr A B
  iscmult : ∀{A B : Set} → Expr A B → Int → Expr A B
  fscmult : ∀{A B : Set} → Expr A B → Float → Expr A Float
  fscmult : ∀{A B : Set} → $(\mathsf{Lan}_{\lambda A\,B.A \times \mathsf{Float}}\ \lambda A\,B.\ \mathsf{Expr}\ A\ B \times \mathsf{Float})\ A\ B$ → Expr A B

- Then Expr can be interpreted as μH for the higher-order functor

$$
\begin{aligned}
H\,F\,X \quad = \quad & \pi_1\,X \\
& + (Lan_{\lambda X\,Y.X \times Int}\ \lambda X\,Y.\,Int)\ X \\
& + (Lan_{\lambda X\,Y.X \times Float}\ \lambda X\,Y.\,Float)\ X \\
& + F\,X\,Y \times F\,X\,Y \\
& + F\,X\,Y \times Int \\
& + (Lan_{\lambda X\,Y.X \times Float}\ \lambda X\,Y.\,F\,X\,Y \times Float)\ X
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 & & + \, F\,X\,Y \times F\,X\,Y \\
 & & + \, F\,X\,Y \times Int \\
 & & + \, (Lan_{\lambda X\,Y. X \times Float}\ \lambda X\,Y.\, F\,X\,Y \times Float)\ X
\end{array}
$$

Completion Choices

- At the level of objects, this gives (at least) the syntactic data elements for GADTs.
- But what about morphisms? What about natural transformations?
- There are two obvious choices:

  - The discrete category $|\mathcal{C}|$ — equivalently, the discrete category $\mathcal{I}$ of interpretations of types in $\mathcal{C}$.

  - The full category $\mathcal{C}$.

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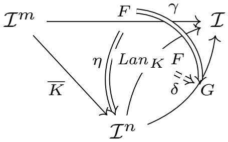

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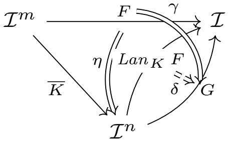

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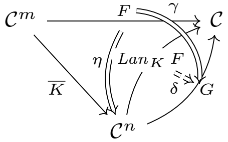

Simplifying Assumptions

- All of the type arguments of our GADT are treated uniformly.
- All of that GADT's data constructors are treated uniformly.
- So we assume for now that a GADT of interest takes exactly one type argument (so $m = n = 1$) and has exactly one data constructor.
- That is, we assume our GADT has the form

$$\begin{aligned}&\mathsf{data\ G\ \ : Set \to Set\ where}\\ &\quad \mathsf{c} : \forall\{\mathsf{A} : \mathsf{Set}\} \to \mathsf{F\,A} \to \mathsf{G\,(K\,A)}\end{aligned}$$

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Computing Left Kan Extensions

- Under the same conditions needed to compute fixpoints of functors using the TFCA, we can compute left Kan extensions using the following well-known colimit formula:

  If $\mathcal{C}$ is locally λ-presentable and F and K are λ-cocontinuous functors on $\mathcal{C}$, then The left Kan extension of F along K can be computed as the colimit

  $$(Lan_K\, F)\, X \;=\; \varinjlim_{(A:\mathcal{C}_0,\, f:KA \to X)} FA$$

- $\mathcal{C}_0$ is a set of objects in $\mathcal{C}$ from which all others can be generated by colimits.
- The idea is that, under these conditions, the "large" colimit that is a left Kan extension can actually be computed as a colimit over a "small" set of support.

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Computing Discrete GADT Interpretations

- In Set:

$$(Lan_K\, F)\, X \;=\; \varinjlim_{A:\mathcal{I},\, f:KA\to X} FA \;=\; \Big(\bigcup_{A:\mathcal{I},\, f:KA\to X} FA\Big)/\sim$$

- Elements of the union are triples $(A : \mathcal{I}, f : KA \to X, y : FA)$ and $\sim$ is the smallest equivalence relation generated by

  $(A, f, y) \sim (A', f', y')$ iff $\exists h : A \to A'$ such that $f = Kh \circ f'$ and $y' = Fhy$

- In the discrete setting, $f = f' = h = id$, so $A = A'$ and $y = y'$, so

$$(Lan_K\, F)\, X \quad = \quad \bigcup_{A:\mathcal{I},\, KA=X} FA$$

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$$(Lan_K\, F)\, (KA) \quad = \quad \{y : FA \mid A : \mathcal{I}\}$$

- GADT syntax gives an element $c\, y$ of $G\, (KA)$ for every $y : FA$.
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$$\begin{array}{ccc} KA & \xrightarrow{\;Kh\;} & KA' \\ & {\scriptstyle f}\searrow \quad \swarrow{\scriptstyle f'} & \\ & X & \end{array}$$

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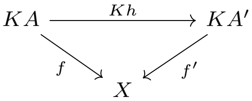

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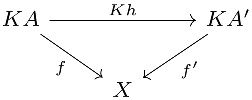

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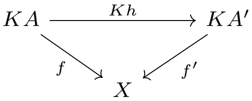

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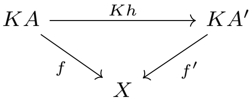

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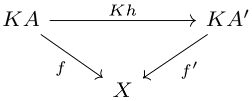

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$$(Lan_K\, F)\, X \;=\; \bigcup_{A:\mathcal{I},\, KA=X} FA$$

i.e.,

$$(Lan_K\, F)\, (KA) \;=\; \{y : FA \mid A : \mathcal{I}\}$$

- GADT syntax gives an element $c\, y$ of $G\,(KA)$ for every $y : FA$.
- Moreover, every element in $G\,(KA)$ is obtained in this way, and instances of G not of the form KA are not inhabited.
- Discrete GADT interpretations contain exactly that data constructed from syntax.

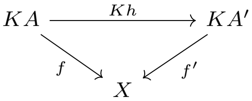
Computing Discrete GADT Interpretations

- In Set:

$$(Lan_K\, F)\, X \;=\; \varinjlim_{A:\mathcal{I},\, f:KA\to X} FA \;=\; \Big(\bigcup_{A:\mathcal{I},\, f:KA\to X} FA\Big)/\sim$$

- Elements of the union are triples $(A:\mathcal{I}, f:KA\to X, y:FA)$ and $\sim$ is the smallest equivalence relation generated by

$(A,f,y)\sim(A',f',y')$ iff $\exists h: A\to A'$ such that $f = Kh\circ f'$ and $y' = Fhy$



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Something Amiss?

- This clearly "works".
- But ADTs and nested types don't need to invoke discreteness to get functoriality.
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- Question: Can we see GADTs as fixpoints of non-discrete functors? That is, can we see GADTs as data types in the "normal", container-y sense, with proper map functions?

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Computing Fully Functorial Interpretations of GADTs

- In Set:

$$(Lan_K\, F)\, X \;=\; \varinjlim_{A:\mathcal{C}_0,\, f:KA\to X} FA \;=\; \Big(\bigcup_{A:\mathcal{C}_0,\, f:KA\to X} FA\Big)/\sim$$

- It is now harder to compute and mod out by the equivalence generated by

  $(A, f, y) \sim (A', f', y')$ iff $\exists h : A \to A'$ such that $f = Kh \circ f'$ and $y' = Fhy$

- Need restrictions on syntax to ensure functoriality of interpretation G of G:
  - Assume GADTs are (hereditarily) polynomial
  - Require strict positivity
  - No truly nested GADTs (no nested Gs in constructor domains or codomains)
- If F and K are higher-order functors then so is $Lan_K\, F$. So $G = \mu J.Lan_K\, F$ is a functor and thus has an associated function map_G.
- Each triple $(A : \mathcal{C}_0, f : KA \to X, y : FA)$ gives an element $map_G\, f\, (c\, y)$ of GX.
- This is cannot possibly be the interpretation of any term constructed from G's syntax unless $X = KB$ for some B.

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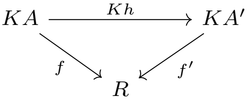

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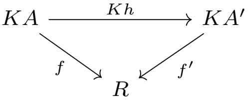

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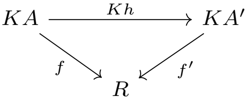

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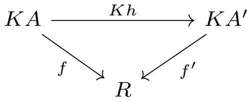

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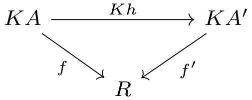

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The Functorial Interpretation of Seq

- We interpret spair : $\forall\{\mathsf{A}\,\mathsf{B} : \mathsf{Set}\} \to \mathsf{Seq}\,\mathsf{A} \to \mathsf{Seq}\,\mathsf{B} \to \mathsf{Seq}(\mathsf{A} \times \mathsf{B})$ as a morphism

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- It is no coincidence that the morphism that $f : A \times B \to X$ that was missing from Seq's map, and thus motivated its discrete semantics, appears in this colimit!
- Thus

$$map_{Seq}\,f\;(spair\;t_1\;t_2)$$

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- The properly functorial interpretation of Seq thus contains data elements not constructed from its syntax!
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Summary

- We have seen that the discrete and fully functorial interpretations of GADTs can be very different.
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- For ADTs and nested types there is only one natural semantics. For GADTs there are two, and they don't even contain the same data elements.

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