Algebraic Theories and Control Effects, Back and Forth

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2014

an example of
the unreasonable effectiveness of mathematics
in computer science and logic

LOLA 2018
Oxford, UK
7.VII.2018

Universal Algebra and Computational Effects

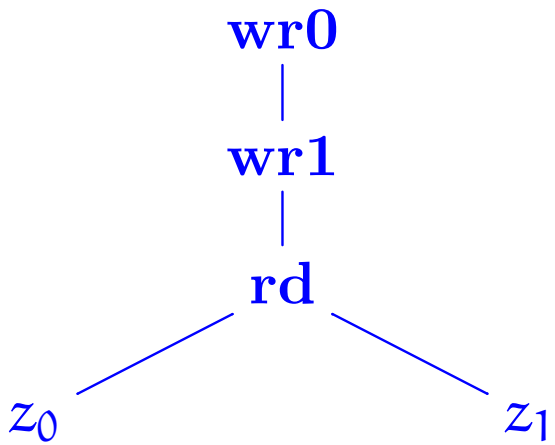
Universal Algebra

[Birkhoff (1935)]

- Signatures

  $\mathbf{wr0} : 1\ ,\ \mathbf{wr1} : 1\ ,\ \mathbf{rd} : 2$

- Free algebras



Universal Algebra

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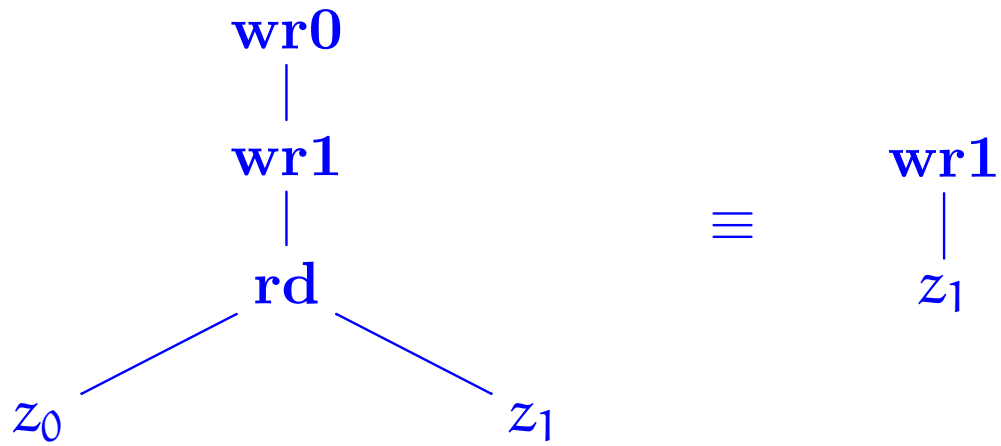


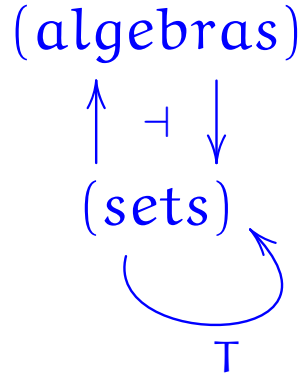
- Equational theories

$$\mathbf{wr0}\big(\mathbf{wr1}(z)\big) \equiv \mathbf{wr1}(z) \quad , \quad \mathbf{wr1}\big(\mathbf{wr0}(z)\big) \equiv \mathbf{wr0}(z)$$
$$\mathbf{wr0}\big(\mathbf{rd}(z_0, z_1)\big) \equiv \mathbf{wr0}(z_0) \quad , \quad \mathbf{wr1}\big(\mathbf{rd}(z_0, z_1)\big) \equiv \mathbf{wr1}(z_1)$$

Notions of Computation

[Moggi (1990); Plotkin, Power (2002)]

- Free-algebra monads



- Computational metalanguage

$$\frac{\Gamma \vdash_c t : \tau}{\Gamma \vdash_c \mathbf{wr0}_\tau(t) : \tau} \qquad \frac{\Gamma \vdash_c t : \tau}{\Gamma \vdash_c \mathbf{wr1}_\tau(t) : \tau}$$

$$\frac{\Gamma \vdash_c t_0 : \tau \qquad \Gamma \vdash_c t_1 : \tau}{\Gamma \vdash_c \mathbf{rd}_\tau(t_0, t_1) : \tau}$$

- Denotational semantics

$$[\![\, \Gamma \vdash_c t : \tau \,]\!] : [\![\, \Gamma \,]\!] \to T[\![\, \tau \,]\!]$$

- $[\![\, \mathbf{wr0}_\tau(t) \,]\!] = [\![\, \Gamma \,]\!] \xrightarrow{[\![t]\!]} T[\![\, \tau \,]\!] \xrightarrow{[\![\mathbf{wr0}]\!]} T[\![\, \tau \,]\!]$
- $[\![\, \mathbf{wr1}_\tau(t) \,]\!] = [\![\, \Gamma \,]\!] \xrightarrow{[\![t]\!]} T[\![\, \tau \,]\!] \xrightarrow{[\![\mathbf{wr1}]\!]} T[\![\, \tau \,]\!]$
- $[\![\, \mathbf{rd}_\tau(t_0, t_1) \,]\!] = [\![\, \Gamma \,]\!] \xrightarrow{\langle [\![t_0]\!], [\![t_1]\!] \rangle} T[\![\, \tau \,]\!] \times T[\![\, \tau \,]\!] \xrightarrow{[\![\mathbf{rd}]\!]} T[\![\, \tau \,]\!]$

- Equational theory
  - $\mathbf{let}\ x = \mathbf{wr0}_\sigma(s)\ \mathbf{in}\ t[x] \equiv \mathbf{wr0}_\tau\big(\mathbf{let}\ x = s\ \mathbf{in}\ t[x]\big)$
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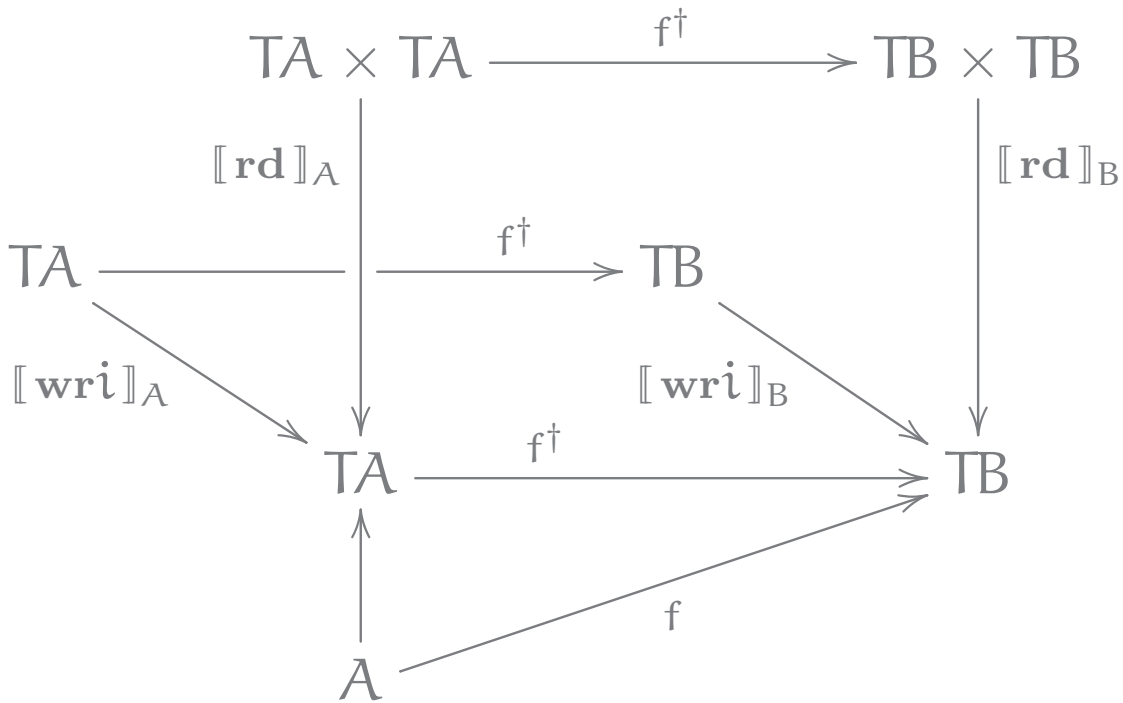

Second-Order Algebraic Theories

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- Binding signatures [Aczel (1978)]

  $\mathbf{app} : (0, 0)\ ,\ \mathbf{abs} : (1)$

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- Free algebras [Fiore, Plotkin, Turi (1999); Hamana (2004); Fiore (2008)]

  $t \ ::= \ \underline{x} \mid M[t_1, \ldots, t_n]$ $\rightsquigarrow$ substitution structure

  $\quad \mid \ \mathbf{app}(t_1, t_2) \mid \mathbf{abs}(x.t')$

[Fiore, Hur (2010); Fiore, Mahmoud (2010)]

- Second-order equational theories

$$(\beta) \quad \mathbf{app}\big(\,\mathbf{abs}(x.\,\mathsf{M}[\underline{x}])\,,\,\mathsf{N}[]\,\big) \;\equiv\; \mathsf{M}\big[\mathsf{N}[]\big]$$

$$(\eta) \quad \mathbf{abs}\big(\,x.\,\mathbf{app}(\,\mathsf{F}[]\,,\,\underline{x}\,)\,\big) \;\equiv\; \mathsf{F}[]$$

[Fiore, Hur (2010); Fiore, Mahmoud (2010)]

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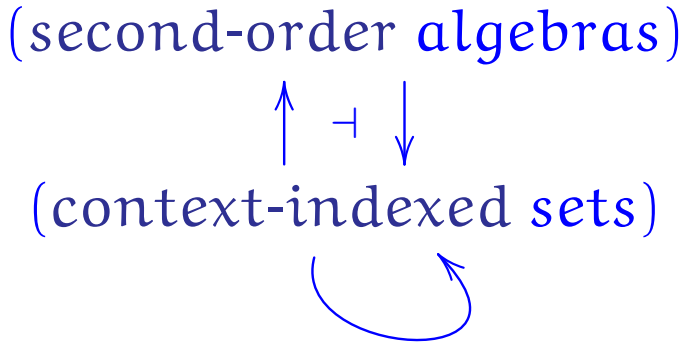
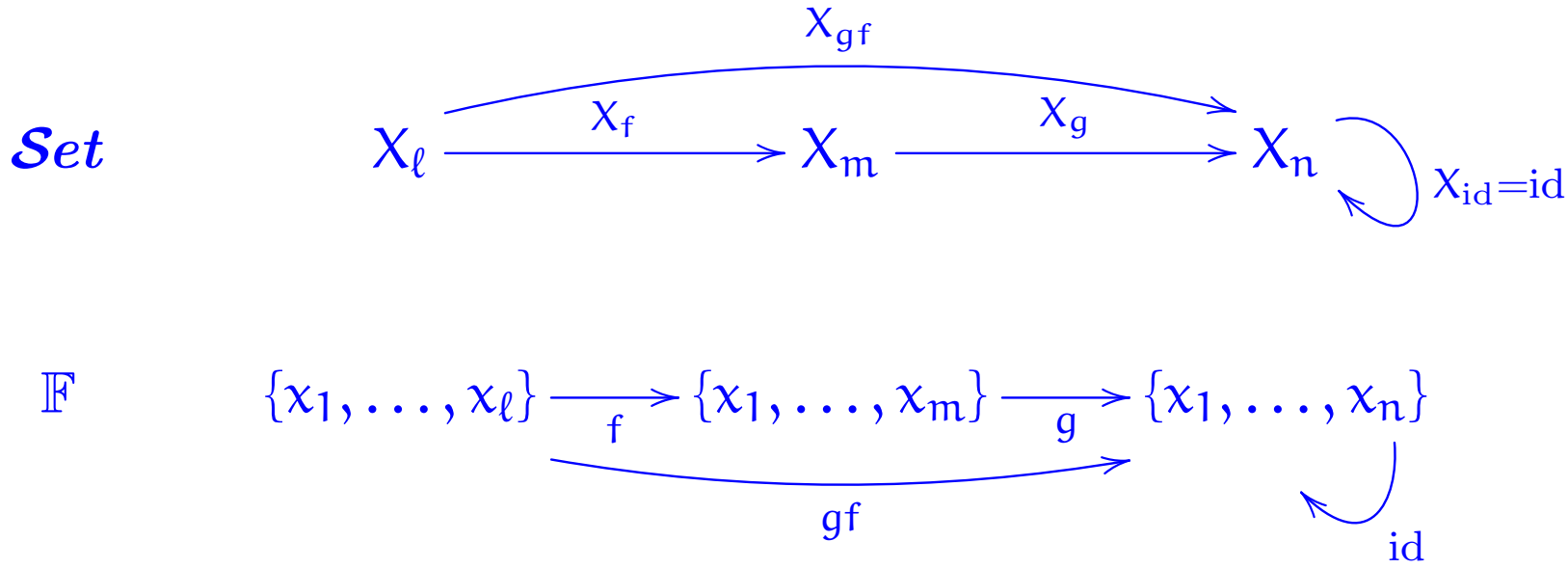
- Free-algebra monads



$\mathcal{S}et^{\mathbb{F}}$

- Context-indexed sets



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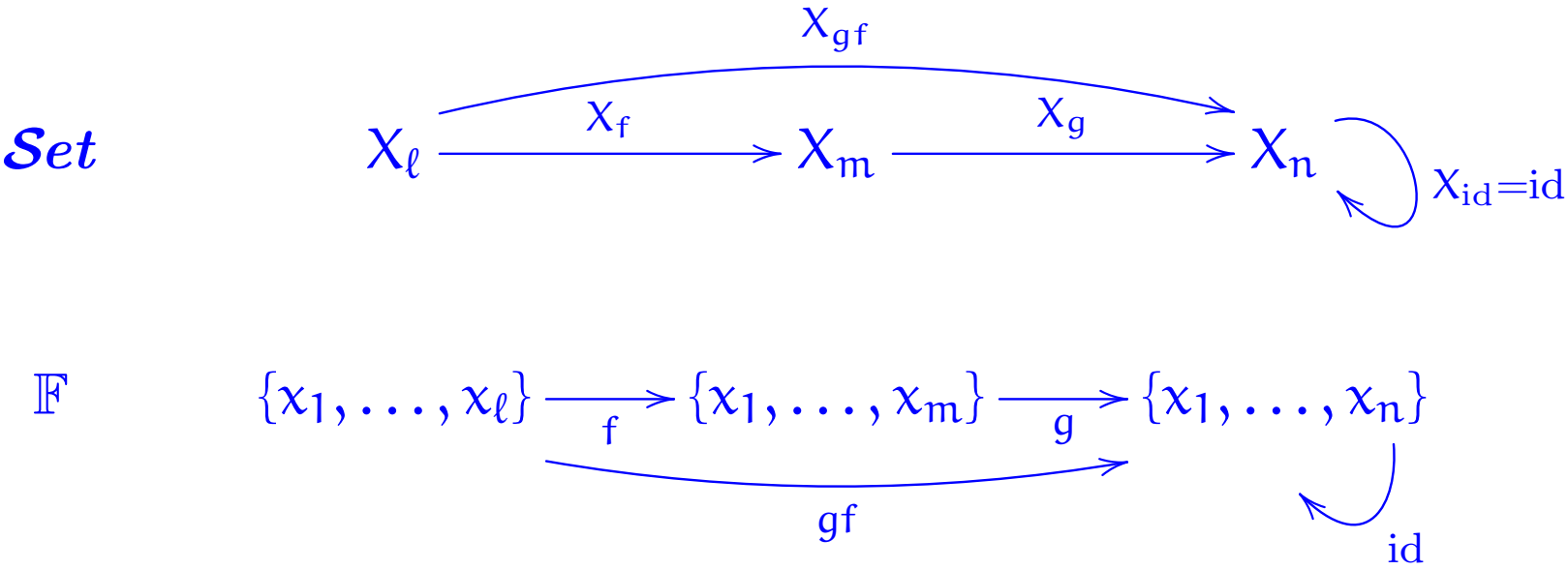
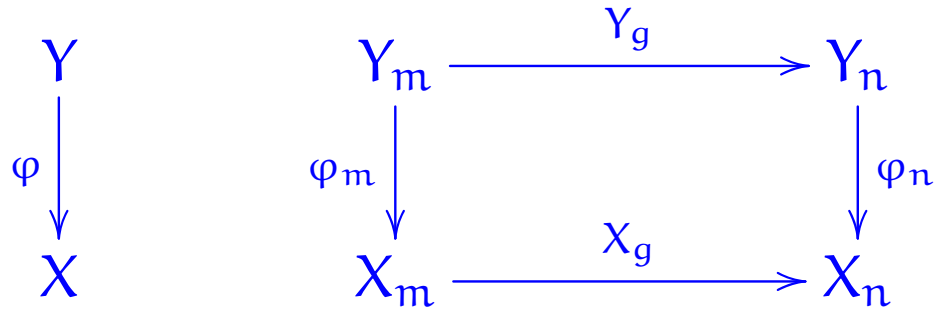


★ **Examples**

$\vec{S}$ $\quad S \rightrightarrows S \rightrightarrows \cdots \qquad S \qquad \cdots$

V $\quad \emptyset \rightarrow \{x_1\} \rightrightarrows \cdots \qquad \{x_1,\ldots,x_n\} \qquad \cdots$

- Renaming-invariant functions



$$\{x_1, \ldots, x_m\} \xrightarrow[g]{} \{x_1, \ldots, x_n\}$$

- Renaming-invariant functions

$$\begin{array}{ccc} Y & & Y_m \xrightarrow{Y_g} Y_n \\ \varphi \downarrow & & \varphi_m \downarrow \quad\quad \downarrow \varphi_n \\ X & & X_m \xrightarrow{X_g} X_n \end{array}$$

$$\{x_1, \ldots, x_m\} \xrightarrow[g]{} \{x_1, \ldots, x_n\}$$

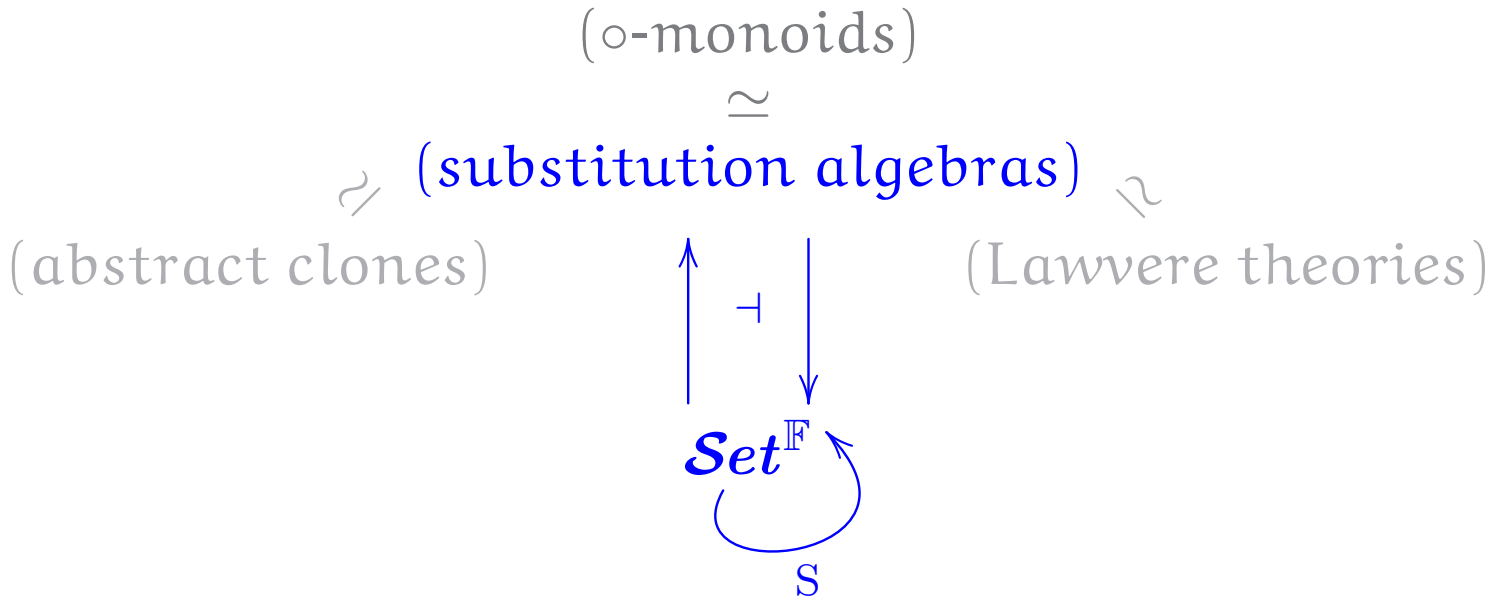
- **NB**

$$\begin{array}{ccccccc} 1 & \{*\} & = & \{*\} & = & \cdots & \{*\} & \cdots \\ \not\downarrow & & & & & & & \\ V & \emptyset & \longrightarrow & \{x_1\} & \rightrightarrows & \cdots & \{x_1, \ldots, x_n\} & \cdots \\ \not\downarrow & & & & & & & \\ 0 & \emptyset & = & \emptyset & = & \cdots & \emptyset & \cdots \end{array}$$

Substitution Algebras

Substitution Algebras [Fiore, Plotkin, Turi (1999)]

- Free substitution-algebra monad



$$SA = \mu X.\, V + A \circ X$$

Substitution Algebras [Fiore, Plotkin, Turi (1999)]

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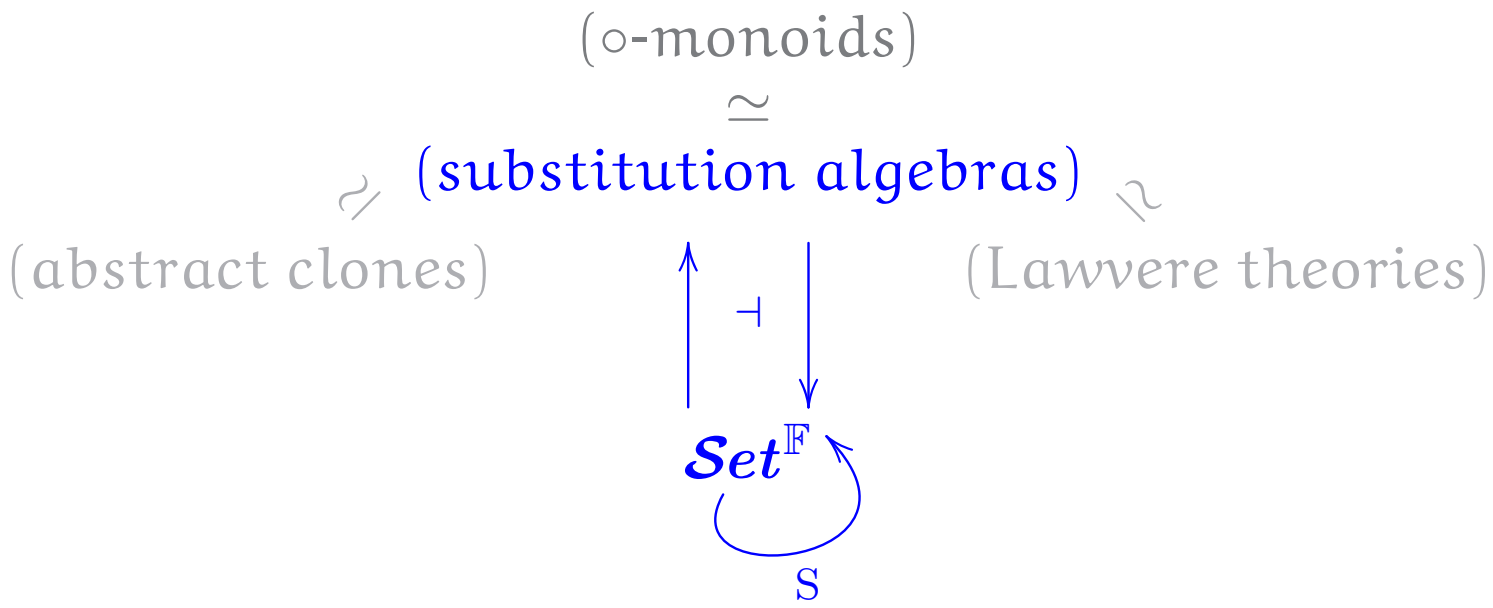

$$SA = \mu X.\, V + A \circ X$$

★ **Definition**

A *substitution algebra* is a structure

$$V \xrightarrow{\textbf{var}} A \xleftarrow{\textbf{sub}} A^V \times A \quad \text{in } \mathcal{S}et^{\mathbb{F}}$$

subject to four axioms: substitution, weakening, extensionality, and associativity.

▸ Substitution Algebra Axioms

Computational Interpretation of Substitution Algebras

Computational Interpretation

[Fiore, Staton (2014)]

- Computational metalanguage

$$\frac{\Gamma \vdash_{\mathsf{v}} e : \mathbf{V}}{\Gamma \vdash_{\mathsf{c}} \mathbf{var}_{\tau}(e) : \tau} \qquad \frac{\Gamma, a : \mathbf{V} \vdash_{\mathsf{c}} t : \tau \quad \Gamma \vdash_{\mathsf{c}} u : \tau}{\Gamma \vdash_{\mathsf{c}} \mathbf{sub}(a.\, t, u) : \tau}$$

Computational Interpretation

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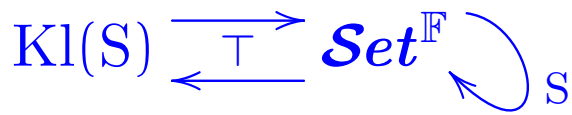
- Denotational semantics

computations values



- Equational theory (I)
  - $\mathbf{let}\ x = \mathbf{sub}\big(\, a.\, \mathsf{M}[a]\,,\ \mathsf{N}[\,]\,\big)\ \mathbf{in}\ \mathsf{P}[x]$

    $\equiv\ \mathbf{sub}\big(\, a.\, \mathbf{let}\ x = \mathsf{M}[a]\ \mathbf{in}\ \mathsf{P}[x]\ \ ,\ \mathbf{let}\ x = \mathsf{N}[\,]\ \mathbf{in}\ \mathsf{P}[x]\ \,\big)$

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  - Weakening axiom

    $\mathbf{sub}\big(\, a.\, \mathsf{M}[\,]\,,\ \mathsf{N}[\,]\,\big) \equiv \mathsf{M}[\,]$

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  - Substitution axiom

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  - Substitution axiom

    $\mathbf{sub}\big(\, a.\, \mathbf{var}(a)\,,\ \mathsf{N}[\,]\,\big) \equiv \mathsf{N}[\,]$
  - Abort law

    $\big(\,\mathbf{let}\ x = \mathbf{var}(a)\ \mathbf{in}\ \mathsf{P}[x]\,\big)\ \equiv\ \mathbf{var}(a)$

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    $\equiv\ \mathbf{sub}\big(\,\mathrm{a}.\,\mathbf{let}\ \mathrm{x} = \mathsf{M}[\mathrm{a}]\ \mathbf{in}\ \mathsf{P}[\mathrm{x}]\ \,,\,\mathbf{let}\ \mathrm{x} = \mathsf{N}[\,]\ \mathbf{in}\ \mathsf{P}[\mathrm{x}]\ \,\big)$
  - Weakening axiom
    $\mathbf{sub}\big(\,\mathrm{a}.\,\mathsf{M}[\,]\,,\,\mathsf{N}[\,]\,\big) \equiv \mathsf{M}[\,]$
  - Substitution axiom
    $\mathbf{sub}\big(\,\mathrm{a}.\,\mathbf{var}(\mathrm{a})\,,\,\mathsf{N}[\,]\,\big) \equiv \mathsf{N}[\,]$
  - Abort law
    $\big(\,\mathbf{let}\ \mathrm{x} = \mathbf{var}(\mathrm{a})\ \mathbf{in}\ \mathsf{P}[\mathrm{x}]\,\big)\ \equiv\ \mathbf{var}(\mathrm{a})$
- Operational intuition

  In $\mathbf{sub}(\mathrm{a}.\mathrm{t}[\mathrm{a}], \mathrm{u})$ the *jump point* a is declared and the computation proceeds as in $\mathrm{t}[\mathrm{a}]$ leading to a value if this produces one, or *aborting and restarting* with the computation of u if $\mathbf{var}(\mathrm{a})$ is invoked.

- Equational theory (II)
  - Extensionality axiom

    $$\mathbf{sub}\big(\,\mathrm{a}.\,\mathsf{M}[\mathrm{a}]\,,\,\mathbf{var}(\mathrm{b})\,\big)\;\equiv\;\mathsf{M}[\mathrm{b}]$$

  - Associativity axiom

    $$\mathbf{sub}\big(\,\mathrm{a}.\,\mathbf{sub}\big(\,\mathrm{b}.\,\mathsf{L}[\mathrm{a},\mathrm{b}]\,,\,\mathsf{M}[\mathrm{a}]\,\big)\,,\,\mathsf{N}[\,]\,\big)$$
    $$\equiv\;\mathbf{sub}\big(\,\mathrm{b}.\,\mathbf{sub}(\,\mathrm{a}.\,\mathsf{L}[\mathrm{a},\mathrm{b}]\,,\,\mathsf{N}[\,]\,)\,,\,\mathbf{sub}(\,\mathrm{a}.\,\mathsf{M}[\mathrm{a}]\,,\,\mathsf{N}[\,]\,)\,\big)$$

★ Substitution, Jumps, and Algebraic Effects

[Fiore, Staton (2014)]

- Computational interpretation of substitution algebras as a code-jumping mechanism.
- Adequate denotational semantics, equational theory, and abstract machine.
- Representation of first-order virtual effects.
- Extension with effect handlers (by structural recursion).

CPS Interpretation of Substitution Algebras

Algebraic CPS Semantics

★ **Theorem** [Lawvere (1969), Kock (1970)]

In the context of strong monads,

$$\frac{\mathrm{T}\text{-algebra structures on objects } A}{\text{monad morphisms } \mathrm{T} \to \mathrm{K}_A}$$

for K_A the double-dualization or continuation monad relative to A.

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for K_A the double-dualization or continuation monad relative to A.

- Algebraic CPS semantics:

$$\frac{\mathrm{V} \text{ is the initial substitution algebra}}{\text{monad morphism } \mathrm{S} \to \mathrm{K}_\mathrm{V}}$$

CPS Interpretation

$$\mathbf{var}_X \longmapsto \lambda a : V.\, \lambda k : V^X.\, a$$

$$\mathbf{sub}_X \longmapsto \lambda\langle M : (K_V X)^V, N : K_V X\rangle.\, \lambda k : V^X.\, M\,(N\,k)\,k$$

$$S \longrightarrow K_V$$

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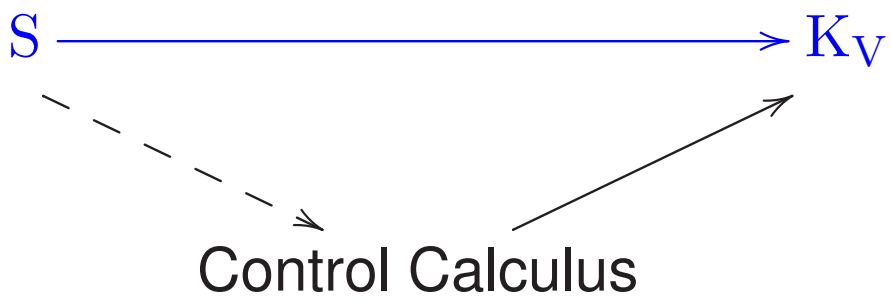

CPS Interpretation in ML

```
signature SubstitutionAlgebra
= sig
    type V
    val var : V -> 'a
    val sub : ( V -> 'a ) * 'a -> 'a
end
```

CPS Interpretation in ML

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signature SubstitutionAlgebra
= sig
    type V
    val var : V -> 'a
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end

structure sa :> SubstitutionAlgebra
= struct
    type R = unit cont
    type V = unit -> R
    fun var( a ) = throw (a()) ()
    fun sub( M , N )
      = callcc( fn k =>  M( fn() => throw k N ) )
  end
```

```
signature ParameterisedSubstitutionAlgebra
= sig
    type 'p V
    val var : 'p V * 'p -> 'a
    val sub : ( 'p V -> 'a ) * ('p -> 'a )  -> 'a
end

structure psa :> ParameterisedSubstitutionAlgebra
= struct
    type R = unit cont
    type 'p V = 'p -> R
    fun var( a , p ) = throw (a p) ()
    fun sub( M , N )
      = callcc( fn k =>  M( fn p => throw k (N p) ) )
  end
```

Inception Algebras

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★ **Definition**

An (L, P)-*inception algebra* is a structure

$$L \times P \xrightarrow{\textbf{recall}} A \xleftarrow{\textbf{incept}} A^L \times A^P$$

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An (L, P)-*inception algebra* is a structure

$$\mathsf{L} \times \mathsf{P} \xrightarrow{\textbf{recall}} A \xleftarrow{\textbf{incept}} A^{\mathsf{L}} \times A^{\mathsf{P}}$$

subject to:

- Substitution axiom

  $$\mathbf{inc}\big(\, \ell.\, \mathbf{rec}(\ell, \mathsf{P}[\,])\,,\; x.\, \mathsf{L}[x] \,\big) \;\equiv\; \mathsf{L}\big[\mathsf{P}[\,]\big]$$

- Extensionality axiom

  $$\mathbf{inc}\big(\, \ell.\, \mathsf{P}[\ell]\,,\; x.\, \mathbf{rec}(k, x) \,\big) \;\equiv\; \mathsf{P}[k]$$

- Weakening axiom

  $$\mathbf{inc}\big(\, \ell.\, \mathsf{M}[\,]\,,\; x.\, \mathsf{L}[x] \,\big) \;\equiv\; \mathsf{M}[\,]$$

- Associativity axiom

  $$\mathbf{inc}\big(\, \ell.\, \mathbf{inc}\big(\, k.\, \mathsf{P}[\ell, k]\,,\; x.\, \mathsf{K}[\ell, x] \,\big)\,,\; y.\, \mathsf{L}[y] \,\big)$$
  $$\equiv\; \mathbf{inc}\big(\, k.\, \mathbf{inc}\big(\, \ell.\, \mathsf{P}[\ell, k]\,,\; y.\, \mathsf{L}[y] \,\big)\,,\; x.\, \mathbf{inc}\big(\, \ell.\, \mathsf{K}[\ell, x]\,,\; y.\, \mathsf{L}[y] \,\big) \,\big)$$

★ Examples

- $(\mathrm{V}, 1)$-inception algebras
  = substitution algebras ($\simeq \circ$-monoids)

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- $(\mathrm{V}, 1)$-inception algebras
  = substitution algebras ($\simeq \circ$-monoids)

- For a set D, $(\mathrm{V}, \vec{\mathrm{D}})$-inception algebras model a *code-jumping with data-passing* mechanism:

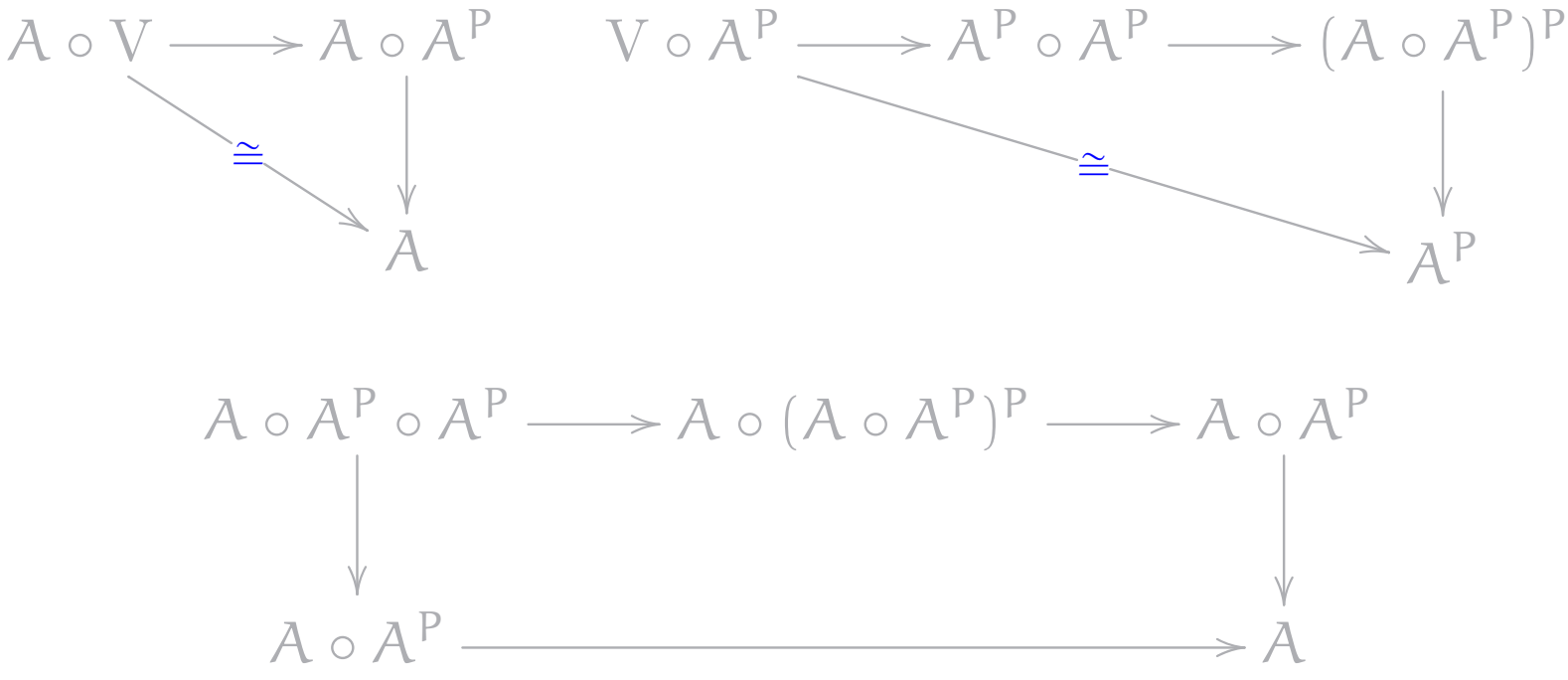
$$\frac{\Gamma \vdash_{\mathsf{v}} e : \mathbf{V} \qquad \Gamma \vdash_{\mathsf{v}} \mathrm{d} : \mathbf{D}}{\Gamma \vdash_{\mathsf{c}} \mathbf{rec}_{\tau}(e, \mathrm{d}) : \tau}$$

$$\frac{\Gamma, \ell : \mathbf{V} \vdash_{\mathsf{c}} \mathrm{t} : \tau \qquad \Gamma, \mathrm{x} : \mathbf{D} \vdash_{\mathsf{c}} \mathrm{u} : \tau}{\Gamma \vdash_{\mathsf{c}} \mathbf{inc}_{\tau}\big(\ell.\,\mathrm{t}\,, \mathrm{x}.\,\mathrm{u}\big) : \tau}$$

★ (V, P)-inception algebras

$\simeq$ unitary Menger algebras of rank P

$$V \to A^P \quad , \quad A \circ A^P \to A$$



★ For a set D, the free $(\mathrm{V}, \vec{\mathrm{D}})$-inception algebra on A is

$$\mu \mathrm{X}.\ (\mathrm{V} \times \vec{\mathrm{D}}) + A \circ \mathrm{X}^{\vec{\mathrm{D}}} \ .$$

($\rightsquigarrow$ effect handlers)

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($\rightsquigarrow$ effect handlers)

- **NB**

  The analysis of free (V, P)-inception algebras suggests new mathematical models and programming primitives.

$$X \bullet Y \ = \ X \circ (\mathrm{V} \times Y)$$

$$\mathbf{inc}_{\mathrm{k}}\big(\ \ell.\,\mathbf{rec}(\ell, \mathsf{P}[\ell])\ ,\ x.\,\mathsf{L}[x]\ \big) \ \equiv \ \mathsf{L}\big[\mathsf{P}[\mathrm{k}]\big]$$

$$\mu X.\ (\mathrm{V} \times \mathrm{P}) + A \bullet X^{\mathrm{P}}$$

Inception Algebras in Logic

- $\mathrm{L} = \neg\,\mathrm{P}$

Negation Elimination

$$\frac{\neg\,\mathrm{P} \qquad \mathrm{P}}{A}$$

Excluded Middle

$$\frac{\begin{array}{cc}[\neg\,\mathrm{P}] & [\mathrm{P}] \\ \vdots & \vdots \\ A & A\end{array}}{A}$$

[de Groote (1995)]

NB Logical inceptions are not ML exceptions in scope.

▸ Counterexample

- Algebraic CPS semantics
  - $(\mathrm{R}^{\mathrm{P}}, \mathrm{P})$-inception algebra structure on R:

$$\begin{cases} \mathbf{rec} & \mapsto & \lambda\langle \ell : \mathrm{R}^{\mathrm{P}}, \mathrm{p} : \mathrm{P} \rangle . \, \ell \, \mathrm{p} \\ \mathbf{inc} & \mapsto & \lambda\langle \mathrm{M} : \mathrm{R}^{(\mathrm{R}^{\mathrm{P}})}, \mathrm{N} : \mathrm{R}^{\mathrm{P}} \rangle . \, \mathrm{M} \, \mathrm{N} \end{cases}$$

- Algebraic CPS semantics
  - (R^P, P)-inception algebra structure on R:

$$\begin{cases} \mathbf{rec} & \mapsto & \lambda\langle \ell : R^P, p : P \rangle.\, \ell\, p \\ \mathbf{inc} & \mapsto & \lambda\langle M : R^{(R^P)}, N : R^P \rangle.\, M\, N \end{cases}$$

  - Induced (R^P, P)-inception algebra structure on $K_R X$:

$$\begin{cases} \mathbf{rec}_X & \mapsto & \lambda\langle \ell : R^P, p : P \rangle.\, \lambda k : R^X.\, \ell\, p \\ \mathbf{inc}_X & \mapsto & \lambda\langle M : (K_R X)^{(R^P)}, N : (K_R X)^P \rangle.\, \lambda k : R^X. \\ & & \quad M\, (\lambda p : P.\, N\, p\, k)\, k \end{cases}$$

- Programming idiom

```
signature LogicalInceptionAlgebra
= sig
    type 'p linc
    val rec : 'p linc * 'p -> 'a
    val inc : ( 'p linc -> 'a ) * ( 'p -> 'a )  -> 'a
  end
```

‣ Logical Inception Algebra Structure

- Programming idiom

```
signature LogicalInceptionAlgebra
= sig
    type 'p linc
    val rec : 'p linc * 'p -> 'a
    val inc : ( 'p linc -> 'a ) * ( 'p -> 'a )  -> 'a
  end
```

- **NB** `inc` generalises and introduces a level of abstraction over `callcc`:

$$\texttt{callcc}(\texttt{f}) \;=\; \texttt{inc}(\texttt{f}, \texttt{id})$$

$$\texttt{inc}(\texttt{f}, \texttt{h}) \;=\; \texttt{callcc}(\texttt{f} \circ \neg\, \texttt{h})$$

- Programming idiom

```
signature LogicalInceptionAlgebra
= sig
    type 'p linc
    val rec : 'p linc * 'p -> 'a
    val inc : ( 'p linc -> 'a ) * ( 'p -> 'a )  -> 'a
  end
```

- **Applications**
  - Classical logic ▸ De Morgan
  - Encoding of local and global exception mechanisms
  - Safe exception handling in program modules
  - Coroutines

Algebraically,

$$\textbf{CPS} = \textbf{NE} + \textbf{EM}$$

Untyped Inception Algebras

- $(\mathrm{V}, \mathrm{V}^{\mathrm{n}})$-inception algebras

$$\mathrm{V} \times \mathrm{V}^{\mathrm{n}} \xrightarrow{\textbf{rec}} A \xleftarrow{\textbf{inc}} A^{\mathrm{V}} \times A^{\mathrm{V}^{\mathrm{n}}}$$

Untyped Inception Algebras

- $(\mathrm{V}, \mathrm{V}^{\mathrm{n}})$-inception algebras

$$\mathrm{V} \times \mathrm{V}^{\mathrm{n}} \xrightarrow{\mathbf{rec}} A \xleftarrow{\mathbf{inc}} A^{\mathrm{V}} \times A^{\mathrm{V}^{\mathrm{n}}}$$

  model the *untyped* CPS calculus[Appel (1992); Thielecke (1997)].

- Conversion table:

$\mathbf{rec}(\ell, \vec{x})$	$\ell\langle\vec{x}\rangle$
$\mathbf{inc}(\ell.\,\mathsf{P}[\ell]\,,\ \vec{x}.\,\mathsf{L}[\vec{x}]\,)$	$\mathsf{P}[\ell]\,\{\,\ell\langle\vec{x}\rangle = \mathsf{L}[\vec{x}]\,\}$

Sorted Inception Algebras

- Sorted sets [Milner (1991)]

$$|-| : S \rightharpoonup S^*$$

- Structures

$$V_\sigma \times \prod_i V_{|\sigma|_i} \qquad A^{V_\sigma} \times A^{\prod_i V_{|\sigma|_i}}$$

$$\searrow \qquad\qquad \swarrow$$

$$A$$

in $\mathcal{S}et^{\mathbb{F}[S]}$

subject to the inception algebra axioms. ▸ Inception Algebra Axioms

★ **Example**

For a set of sorts S, the sorted set

$$T_S \rightharpoonup T_S{}^*, \text{ where } T_S = \mu X.\, S + X^*$$

yields the typed CPS calculus, and hence sorted $\otimes\neg$-categories[Thielecke (1997)].

Back to Control Effects

Some Developments

- Untyped inception algebras
  - Recursion.
  - Algebraic interpretation of the lambda calculus in the untyped CPS calculus.
- Right Lambda Algebras
  - Computational interpretation as a mechanism for stack manipulation of code pointers. [Fiore, Staton (2014)]
  - Stack abstract machine for the untyped CPS calculus, and hence also for the lambda calculus.
- Left Lambda Algebras
  - Computational interpretation as a synchronous coroutine mechanism.
  - Producer-Consumer programming pattern.

Untyped Inception Algebras
Recursion
CPS Lambda Structure
Right Lambda Algebras
Stack Abstract Machine
Left Lambda Algebras
Coroutine Mechanism
Producer-Consumer

Final Remarks

Final Remarks

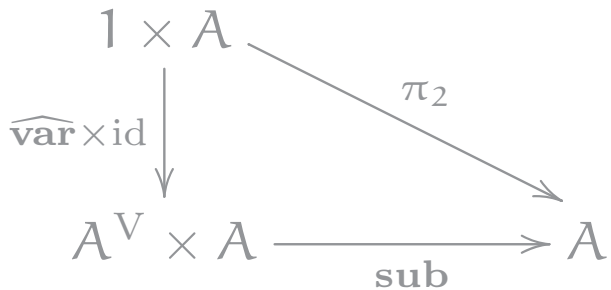
- Conclusions
  - The first algebraic axiomatisation of a variety of control effects.
  - Foundational analysis of the principles of programming with algebraic effects.
  - Inception algebras
- Directions
  - Semantic models
  - Classical/intermediate logics
  - Algebraic theories
  - Programming

Appendix

Substitution Algebra Axioms

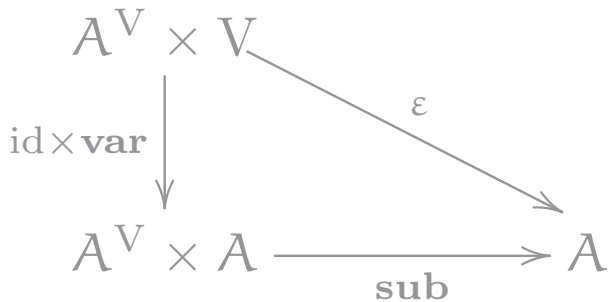
- Substitution

$$\mathbf{sub}\big(\, a.\,\mathbf{var}(a)\,,\, M[\,]\,\big) \;\equiv\; M[\,]$$



- Extensionality

$$\mathbf{sub}\big(\, a.\,M[a]\,,\, \mathbf{var}(b)\,\big) \;\equiv\; M[b]$$

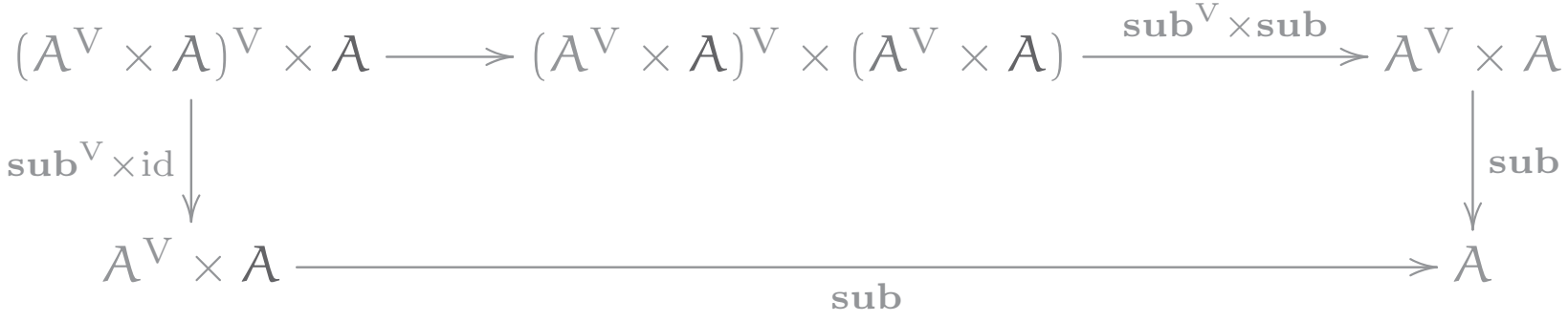


- Weakening

$$\mathbf{sub}\big(\,a.\,M[\,]\,,\,N[\,]\,\big)\;\equiv\;M[\,]$$

- Associativity

$$\mathbf{sub}\big(\,a.\,\mathbf{sub}\big(\,b.\,L[a,b]\,,\,M[a]\,\big)\,,\,N[\,]\,\big)$$
$$\equiv\;\mathbf{sub}\big(\,b.\,\mathbf{sub}(\,a.\,L[a,b]\,,\,N[\,]\,)\,,\,\mathbf{sub}(\,a.\,M[a]\,,\,N[\,]\,)\,\big)$$



Contraction Laws

- $\mathbf{sub}\big(\, a.\, M[a,b]\,,\, \mathbf{var}(b)\,\big) \;\equiv\; M[b,b]$

- $\mathbf{sub}\big(\, a.\, \mathbf{sub}(\, b.\, M[a,b]\,,\, N[\,]\,)\,,\, N[\,]\,\big) \;\equiv\; \mathbf{sub}\big(\, c.\, M[c,c]\,,\, N[\,]\,\big)$

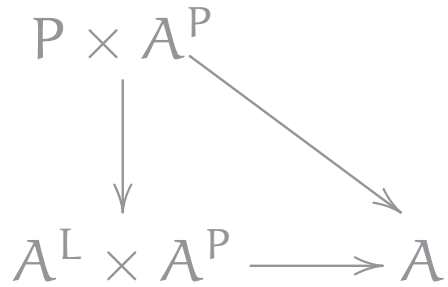
- $\mathbf{sub}\big(\, c.\, \mathbf{let}\; x = M[c]\; \mathbf{in}\; N[x,c]\;\;,\, L[\,]\,\big)$

  $\equiv\;\; \mathbf{sub}\big(\, a.\, \mathbf{let}\; x = M[a]\; \mathbf{in}\; \mathbf{sub}\big(b.\, N[x,b], L[\,]\big)\;\;,\, L[\,]\,\big)$

Inception Algebra Axioms

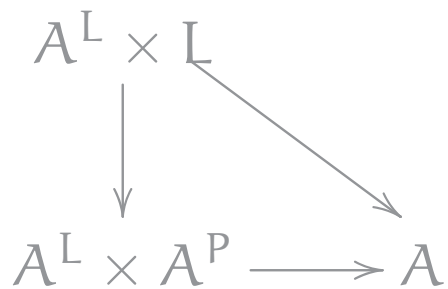
- Substitution

$$\mathbf{inc}\big(\,\ell.\,\mathbf{rec}(\ell, \mathsf{P}[\,])\,,\; x.\,\mathsf{L}[x]\,\big) \;\equiv\; \mathsf{L}[\mathsf{P}]$$



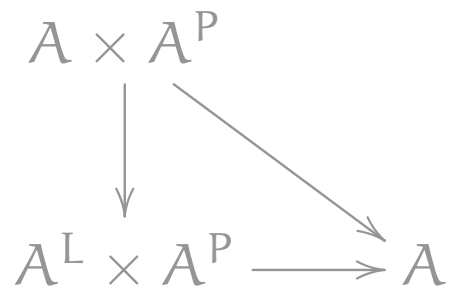
- Extensionality

$$\mathbf{inc}\big(\,\ell.\,\mathsf{P}[\ell]\,,\; x.\,\mathbf{rec}(k, x)\,\big) \;\equiv\; \mathsf{P}[k]$$



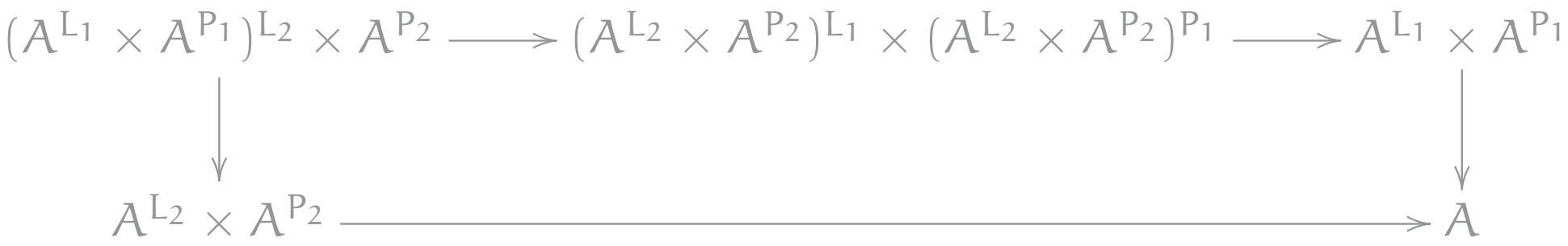
- Weakening

$$\mathbf{inc}\big(\,\ell.\,\mathsf{M}[\,]\,,\,x.\,\mathsf{L}[x]\,\big)\;\equiv\;\mathsf{M}[\,]$$



- Associativity

$$\mathbf{inc}\big(\,\ell.\,\mathbf{inc}\big(\,k.\,\mathsf{P}[\ell,k]\,,\,x.\,\mathsf{K}[\ell,x]\,\big)\,,\,y.\,\mathsf{L}[y]\,\big)$$
$$\equiv\;\mathbf{inc}\big(\,k.\,\mathbf{inc}\big(\,\ell.\,\mathsf{P}[\ell,k]\,,\,y.\,\mathsf{L}[y]\,\big)\,,\,x.\,\mathbf{inc}\big(\,\ell.\,\mathsf{K}[\ell,x]\,,\,y.\,\mathsf{L}[y]\,\big)\,\big)$$



▸ Sorted Inception Algebras

Inceptions Are Not Local Exceptions

- **NB**

$$\mathbf{sub}\big(\, e_1 . \mathbf{let}\ \ell = \mathbf{sub}(\, e_2 . \mathbf{ret}\, e_2 \,,\, \mathbf{ret}\, e_1 \,)\ \mathbf{in}\ \mathbf{var}\, \ell \ ,\, \langle\,\rangle \,\big) \ \equiv\ \langle\,\rangle$$

however

```
let exception e1
in ( let val l
          = let exception e2
            in e2
            handle e2 => e1 end
     in
       raise l
     end )
handle e1 => () end
```

outputs `uncaught exception e2`.

Logical Inception Algebra Structure

```
signature LogicalInceptionAlgebra
= sig
    type 'p linc
    val rec : 'p linc * 'p -> 'a
    val inc : ( 'p linc -> 'a ) * ( 'p -> 'a )  -> 'a
  end

structure lia :> LogicalInceptionAlgebra
= struct
    type R = unit cont
    type 'p linc = 'p -> R
    fun rec( a , p ) = throw (a p) ()
    fun inc( M , N )
      = callcc( fn k => M( fn p => throw k (N p) ) )
  end
```

De Morgan

```
val DeMorgan : ( 'a * 'b ) linc -> ('a linc,'b linc) sum
  = fn c =>
    inc( fn a => left a ,
         fn x => inc( fn b => right b ,
                      fn y => rec( c , (x,y) ) ) )
```

Nullary/Unary Untyped Inception Algebras

- Self recall

$$\mathbf{inc}\big(\,\ell.\,\mathbf{rec}(\ell,\ell)\,,\,x.\,\mathsf{M}[x]\,\big)\;\equiv\;\mathbf{inc}\big(\,\ell.\,\mathsf{M}[\ell]\,,\,x.\,\mathsf{M}[x]\,\big)$$

Nullary/Unary Untyped Inception Algebras

- Self recall

$$\mathbf{inc}\big(\,\ell.\,\mathbf{rec}(\ell,\ell)\,,\,x.\,M[x]\,\big)\;\equiv\;\mathbf{inc}\big(\,\ell.\,M[\ell]\,,\,x.\,M[x]\,\big)$$

- Recursion

  For

$$Y_t\;=\;\mathbf{inc}\big(\,\ell.\,\mathbf{rec}(\ell,\ell)\,,\,\ell.\,\mathbf{sub}\big(\,a.\,t[a]\,,\,\mathbf{rec}(\ell,\ell)\big)\,\big)$$

  we have

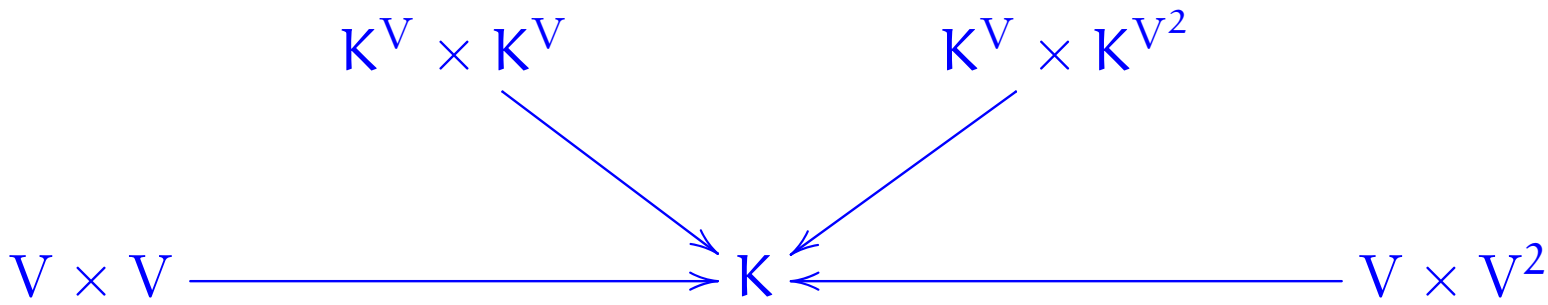
$$Y_t\;\equiv\;\mathbf{sub}\big(\,a.\,t[a]\,,\,Y_t\,\big)$$

▸ Developments

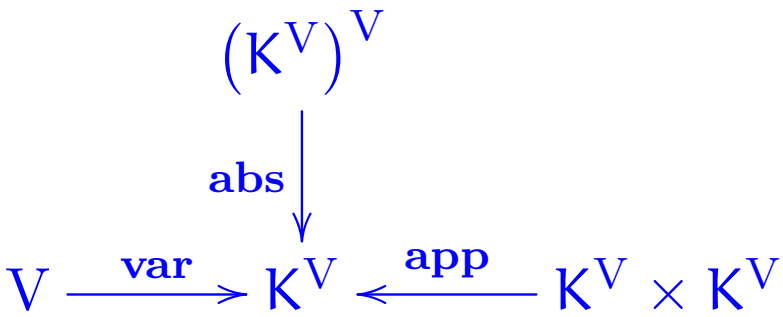
Unary/Binary Untyped Inception Algebras

- CPS lambda structure

The initial unary/binary untyped inception algebra



provides a CPS lambda structure



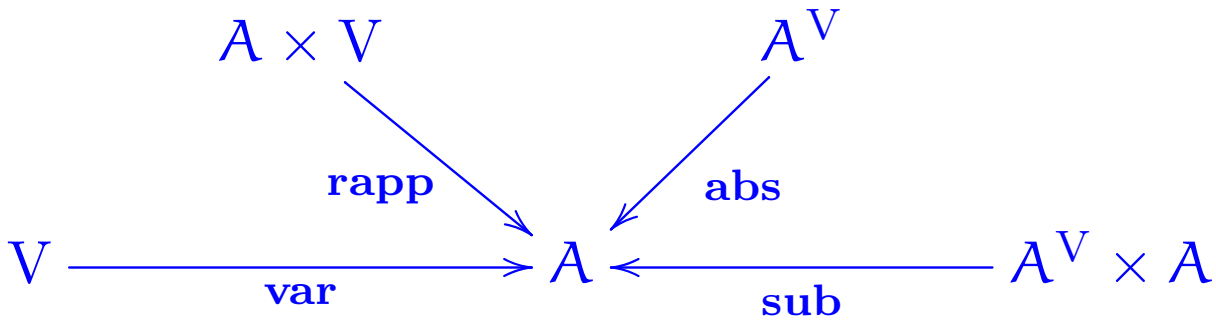
thereby inducing an homomorphic CPS interpretation

$$\Lambda \to \mathrm{K}^{\mathrm{V}}$$

of lambda terms.

- CPS interpretation
  - $\mathbf{var} : \mathrm{x}[e] \mapsto \mathbf{rec}(\, e\,,\, \langle \mathrm{x} \rangle\,)$
  - $\mathbf{abs} : \mathsf{M}[\mathrm{x}][e] \mapsto \mathbf{inc}(\;\; \mathrm{f}.\mathbf{rec}(\, e\,,\, \langle \mathrm{f} \rangle\,)\;\;,\;\; \langle \mathrm{a}, e \rangle.\, \mathsf{M}[\mathrm{a}][e]\;\;)$
  - $\mathbf{app} : \mathsf{M}[e], \mathsf{N}[e] \mapsto \mathbf{inc}(\;\; \mathrm{m}.\mathsf{M}[\mathrm{m}]\;\;,\;\; \langle \mathrm{f} \rangle.\, \mathbf{inc}(\;\; \mathrm{n}.\mathsf{N}[\mathrm{n}]\;\;,\;\; \langle \mathrm{a} \rangle.\, \mathbf{rec}(\mathrm{f}, \langle \mathrm{a}, e \rangle)\;\;)\;\;)$

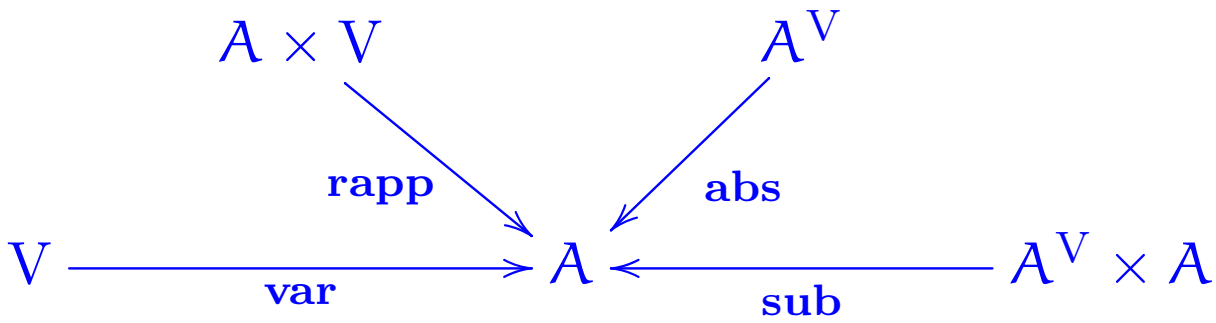
Right Lambda Algebras



$$(\beta_r) \quad \mathbf{rapp}\big(\,\mathbf{abs}(x.\,M[x])\,,\,a\,\big) \;\equiv\; M[a]$$

[Hirschowitz, Maggesi (2010); Hyland (2013)]

Right Lambda Algebras



push a

$$\underbrace{(\beta_r)}_{\text{summon}} \qquad \overbrace{\mathbf{rapp}\big(\underbrace{\mathbf{abs}(x.\,M[x])}_{\text{pop in } x},\, a\big)}^{\text{push } a} \;\equiv\; M[a]$$

- Computational interpretation: [Fiore, Staton (2014)]

  A mechanism for stack manipulation of code pointers.

- Application:

  Stack abstract machine for CPS calculus.

- Stack CPS structure

- $$\begin{array}{rcl} \mathrm{V} \times \mathrm{V}^{n} & \longrightarrow & A \\ a, \langle a_1, \ldots, a_n \rangle & \mapsto & \mathbf{push}(\cdots \mathbf{push}(\mathbf{var}(a), a_1) \cdots, a_n) \end{array}$$

- $$\begin{array}{rcl} A^{\mathrm{V}} \times A^{\mathrm{V}^{n}} & \longrightarrow & A \\ \mathsf{M}[a], \mathsf{N}[a_1, \ldots, a_n] & \mapsto & \mathbf{sub}(\ a.\,\mathsf{M}[a]\ , \\ & & \quad \mathbf{pop}(a_1.\cdots \mathbf{pop}(a_n.\mathsf{N}[a_1, \ldots, a_n]))\ ) \end{array}$$

▸ Developments

- Parameterised fixpoint
  - For

$$\mathsf{T}[\mathrm{f}, \mathrm{x}] = \mathbf{in}(\mathbf{yield}(\mathrm{x}, \mathbf{thunk}(z.\, \mathbf{push}(z, \mathbf{push}(\mathrm{f}, \mathbf{var}\ \mathrm{f})))))$$

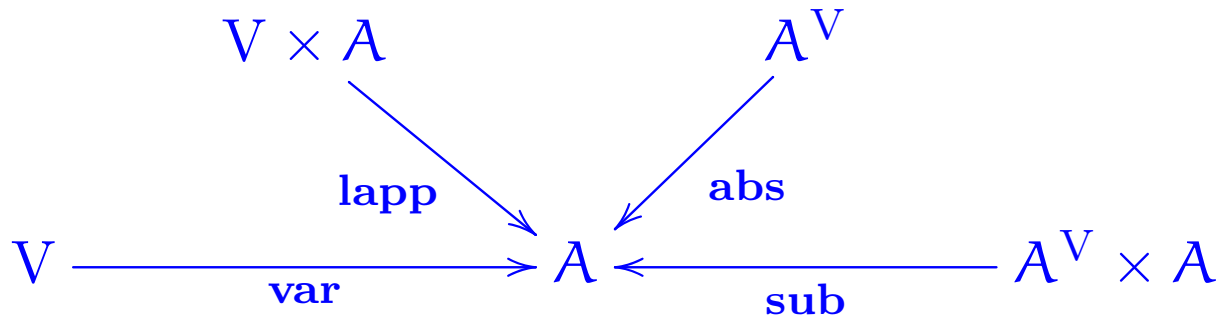
    and

$$\mathbf{In}[\mathrm{x}] = \mathbf{sub}\big(\, \mathrm{f}.\, \mathsf{T}[\mathrm{f}, \mathrm{x}]\,,\ \mathbf{pop}\big(\, \mathrm{f}.\, \mathbf{pop}(\mathrm{x}.\, \mathrm{t}[\mathrm{f}, \mathrm{x}])\,\big)\,\big)$$

    we have

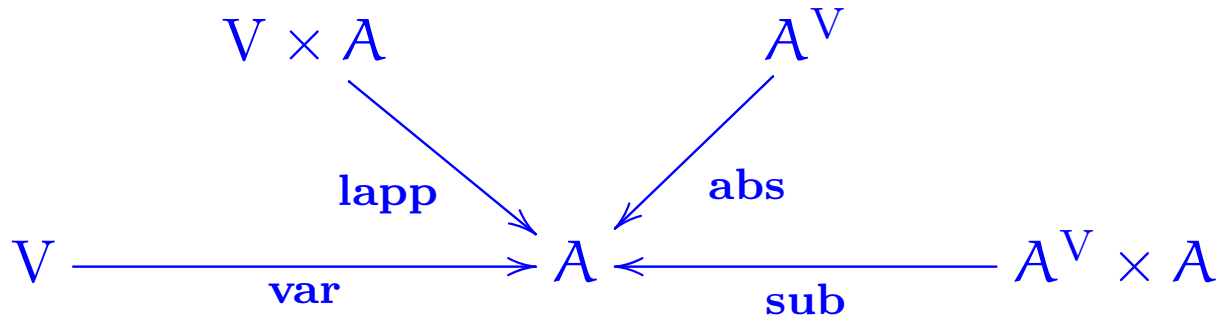
$$\mathbf{In}[\mathrm{x}] \ \equiv\ \mathbf{in}(\mathbf{yield}(\mathrm{x}, \mathbf{thunk}(z.\, \mathbf{In}[z])))$$

Left Lambda Algebras



$$(\beta_l) \quad \mathbf{sub}\big(\,\ell.\,\mathbf{lapp}(\ell, \mathsf{N}[\,])\,,\,\mathbf{abs}(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,)\,\big) \;\equiv\; \mathbf{sub}\big(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,,\,\mathsf{N}[\,]\,\big)$$

Left Lambda Algebras



$$\underbrace{(\beta_l)}_{\text{resume}} \quad \mathbf{sub}\big(\,\ell.\,\underbrace{\mathbf{lapp}(\ell, \mathsf{N}[\,])}_{\text{yield N}},\ \underbrace{\mathbf{abs}(\,\mathrm{k}.\,\mathsf{M}[\mathrm{k}]\,)}_{\text{thunk M}}\,\big) \equiv \mathbf{sub}\big(\,\mathrm{k}.\,\mathsf{M}[\mathrm{k}]\,,\,\mathsf{N}[\,]\,\big)$$

- Computational interpretation:

  A synchronous coroutine mechanism

- Application:

  Producer-Consumer

▸ Producer-Consumer

▸ Developments

- Axioms
  - $\mathbf{sub}\big(\,\ell.\,\mathbf{yield}(\ell, \mathsf{N}[\,])\,,\,\mathbf{thunk}(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,)\,\big) \;\equiv\; \mathbf{sub}\big(\,\mathsf{k}.\,\mathsf{M}[\mathsf{k}]\,,\,\mathsf{N}[\,]\,\big)$
  - $\mathbf{sub}\big(\,\ell.\,\mathbf{thunk}\big(x.\,\mathsf{M}[\ell, x]\big)\,,\,\mathsf{N}[\,]\,\big)$
    $\equiv\;\; \mathbf{thunk}\big(\,x.\,\mathbf{sub}\big(\,\ell.\,\mathsf{M}[\ell, x]\,,\,\mathsf{N}[\,]\,\big)\,\big)$
  - $\mathbf{sub}\big(\,e.\,\mathbf{yield}\big(\mathsf{L}[e], \mathsf{M}[e]\big)\,,\,\mathsf{N}[\,]\,\big)$
    $\equiv\;\; \mathbf{sub}\big(\;\ell.\,\mathbf{yield}\big(\,\ell\,,\,\mathbf{sub}\big(e.\,\mathsf{M}[e], \mathsf{N}[\,]\big)\,\big)\;\;,$
    $\mathbf{sub}\big(\,e.\,\mathbf{var}(\mathsf{L}[e]), \mathsf{N}[\,]\big)\;\;\big)$

- Fixpoints
  - For a computation $\mathtt{t}(\cdot)$,

    $$\mathsf{Y}_\mathtt{t} = \mathbf{sub}\big(\,\mathtt{x}.\,\mathtt{t}\big(\mathbf{yield}(\mathtt{x}, \mathbf{var}\,\mathtt{x})\big)\,,\ \mathbf{thunk}\big(\,\mathtt{x}.\,\mathtt{t}\big(\mathbf{yield}(\mathtt{x}, \mathbf{var}\,\mathtt{x})\big)\,\big)\big)$$

    we have

    $$\mathsf{Y}_\mathtt{t} \equiv \mathtt{t}(\mathsf{Y}_\mathtt{t})$$

  - For a parameterised computation $\mathtt{t}[\mathtt{x}](\mathtt{c}.\mathsf{M}[\mathtt{c}])$, let

    $$\begin{aligned}\mathsf{T} = \ \ & \mathbf{thunk}(\mathtt{f}.\,\mathbf{thunk}(\mathtt{x}.\\ & \quad \mathtt{t}[\mathtt{x}]\big(\,\mathtt{c}.\,\mathbf{sub}(\mathtt{p}.\,\mathbf{yield}(\mathtt{p}, \mathbf{var}\ \mathtt{c}), \mathbf{yield}(\mathtt{f}, \mathbf{var}\ \mathtt{f}))\,\big)))\end{aligned}$$

    For

    $$\mathsf{Y}_\mathtt{t}[\mathtt{x}] = \mathbf{sub}\big(\,\mathtt{f}.\,\mathtt{t}[\mathtt{x}](\mathtt{c}.\,\mathbf{sub}(\mathtt{p}.\mathbf{yield}(\mathtt{p}, \mathbf{var}\ \mathtt{c}), \mathbf{yield}(\mathtt{f}, \mathsf{T}))\,,\ \mathsf{T}\,\big)$$

    we have

    $$\mathsf{Y}_\mathtt{t}[\mathtt{x}] \ \equiv \ \mathtt{t}[\mathtt{x}]\big(\,\mathtt{c}.\,\mathsf{Y}_\mathtt{t}[\mathtt{c}]\,\big)$$

- Producer-Consumer: one in, two out

  Let

$$\begin{aligned}\mathbf{In}[a] &= \mathbf{in}(\mathbf{yield}(a, \mathbf{thunk}(x.\mathbf{In}[x]))) \\ \mathbf{Out} &= \mathbf{thunk}(y.\mathbf{out}(\mathbf{out}(\mathbf{yield}(y, \mathbf{Out}))))\end{aligned}$$

  Then

$$\mathbf{sub}(\, a.\, \mathbf{In}[a]\,,\, \mathbf{Out}\,) \equiv \mathbf{in}(\mathbf{out}(\mathbf{out}(\mathbf{sub}(x.\mathbf{In}[x], \mathbf{Out}))))$$

▸ Developments