A General Framework for Relational Parametricity

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January 2018

Abstract

Reynolds’ original theory of relational parametricity was intended to capture the idea that polymorphically typed System F programs preserve all relations between inputs. But as Reynolds himself later showed, his theory can only be formalized in a meta-theory with an impredicative universe, such as the Calculus of Inductive Constructions. Abstracting from Reynolds’ ideas, Dunphy and Reddy developed their well-known framework for parametricity that uses parametric limits in reflexive graph categories and aims to subsume a variety of parametric models. As we observe, however, their theory is not sufficiently general to subsume the very model that inspired parametricity, namely Reynolds’ original model, expressed inside type theory.

To correct this, we develop an abstract framework for relational parametricity that generalizes the notion of a reflexive graph categories and delivers Reynolds’ model as a direct instance in a natural way. This framework is uniform with respect to a choice of meta-theory, which allows us to obtain the well-known PER model of Longo and Moggi as a direct instance in a natural way as well. In addition, we offer two novel relationally parametric models of System F: i) a categorical version of Reynolds’ model, where types are functorial on isomorphisms and all polymorphic functions respect the functorial action, and ii) a proof-relevant categorical version of Reynolds’ model (after Orsanigo), where, additionally, witnesses of relatedness are themselves suitably related. We show that, unlike previously existing frameworks for parametricity, ours recognizes both of these new models in a natural way. Our framework is thus descriptive, in that it accounts for well-known models, as well as prescriptive, in that it identifies abstract properties that good models of relational parametricity should satisfy and suggests new constructions of such models.

1 Introduction

Reynolds [13] introduced the notion of relational parametricity to model the extensional behavior of programs in System F [6], the formal calculus at the core of all polymorphic functional languages. His goal was to give a type \( \alpha \vdash T(\alpha) \) an object interpretation \( T_0 \) and a relational interpretation \( T_1 \), where \( T_0 \) takes sets to sets and \( T_1 \) takes relations \( R \subseteq A \times B \) to relations \( T_1(R) \subseteq T_0(A) \times T_0(B) \). A term
always mean the version of his model that is internal to extensional CIC as described above. The need for
\( \alpha \) on the structure of \( T \) and \( t \) in such a way that they implied two key theorems: the Identity
Extension Lemma, stating that if \( R \) is the equality relation on \( A \) then \( T_1(R) \) is the
equality relation on \( T_0(A) \); and the Abstraction Theorem, stating that, for any relation
\( R \subseteq A \times B, t_0(A) \) and \( t_0(B) \) map arguments related by \( S_1(R) \) to results related by
\( T_1(R) \). A similar result holds for types and terms with any number of free variables.

In Reynolds’ treatment of relational parametricity, if \( U(\alpha) \) is the type \( \alpha \vdash S(\alpha) \to T(\alpha) \), for example, then \( U_0(A) \) is the set of functions \( f : S_0(A) \to T_0(A) \) and, for
\( R \subseteq A \times B, U_1(R) \) relates \( f : S_0(A) \to T_0(A) \) to \( g : S_0(B) \to T_0(B) \) iff \( f \) and \( g \)
map arguments related by \( S_1(R) \) to results related by \( T_1(R) \). Similarly, if \( V \) is the type
\( \forall \alpha. S(\alpha) \), then \( V_0 \) consists of those polymorphic functions \( f \) that take a set \( A \) and
return an element of \( S_0(A) \), and also have the property that for any relation \( R \subseteq A \times B, f(A) \) and \( f(B) \) are related by \( S_1(R) \). Two such polymorphic functions \( f \) and \( g \) are then
related by \( V_1 \) iff for any relation \( R \subseteq A \times B, f(A) \) and \( g(B) \) are related by \( S_1(R) \).
These definitions allow us to deduce interesting properties of (interpretations of) terms
solely from their types. For example, for any term \( t : \forall \alpha. \alpha \to \alpha \), the Abstraction
Theorem guarantees that the interpretation \( t_0 \) of \( t \) is related to itself by the relational
interpretation of \( \forall \alpha. \alpha \to \alpha \). So if we fix a set \( A, a \in A \), and define a relation on
\( A \) by \( R := \{(a,a)\} \), then \( t_0(A) \) must be related to itself by the relational interpretation of
\( \alpha \vdash \alpha \to \alpha \) applied to \( R \). This means that \( t_0(A) \) must carry arguments related by
\( R \) to results related by \( R \). Since \( a \) is related to itself by \( R \), \( t_0(A) a \) must be related to
itself by \( R \), so that \( t_0(A) a \) must be \( a \). That is, \( t_0 \) must be the polymorphic identity
function. Such applications of relational parametricity are useful in many different
scenarios, e.g., when proving invariance of polymorphic functions under changes of
data representation, equivalences of programs, and “free theorems” [17].

The well-known problem with Reynolds’ treatment of relational parametricity (see
[14]) is that the universe of sets is not impredicative, and hence the aforementioned
“set” \( V_0 \) cannot be formed. This issue can be resolved if we instead work in a meta-
theory that has an impredicative universe; a natural choice is an extensional version of
the Calculus of Inductive Constructions (CIC), i.e., a dependent type theory with a cumu-
lative Russell-style hierarchy of universes \( U_0 : U_1 : \ldots \), where \( U_0 \) is impredicative,
and extensional identity types. With this adjustment, we have two canonical relation-
ally parametric models of System F: i) the PER model of Longo and Moggi [9], internal
to the theory of \( \omega \)-sets and realizable functions, and ii) Reynolds’ original model, internal
to CIC.

After Reynolds’ original paper, more abstract treatments of his ideas were given
by, e.g., Robinson and Rosolini [15], O’Hearn and Tennent [11], Dunphy and Reddy
[2], and Ghani et al. [5]. The approach is to use a categorical structure — reflexive
graph categories for [2, 11, 15] and fibrations for [5] — to represent sets and relations,
and to interpret types as appropriate functors and terms as natural transformations.
In particular, [2] aims to “[address] parametricity in all its incarnations”, and similarly for
[5]. Surprisingly and significantly, however, Reynolds’ original model does not arise
\[1\] Since there are no set-theoretic models of System F, by the phrase “Reynolds’ original model” we will
always mean the version of his model that is internal to extensional CIC as described above. The need for
impredicativity is inherited from Reynolds’ original construction, and is not a new requirement.
as a direct instance of either framework. This leads us to ask:

**What constitutes a good framework for relational parametricity?**

Our answer is that such a framework should:

1. *Deliver a relationally parametric model for each instantiation of its parameters, from which it uniformly produces such models. In particular, it should allow a choice of a suitable meta-theory (the Calculus of Inductive Constructions, the theory of \( \omega \)-sets, etc.).*

2. *Admit the two canonical relationally parametric models mentioned above as direct instances in a natural, uniform way.*

3. *Abstractly formulate properties that good models of parametricity for System F should be expected to satisfy.*

Criterion 1 ensures that we indeed get a true framework rather than just a reusable blueprint for constructing models of parametricity. Criterion 2 remains unsatisfied for the frameworks of Dunphy and Reddy and of Ghani *et al.* because Reynolds’ original model formulated syntactically does not satisfy certain strictness conditions imposed by [2, 5]. For example, let \( \alpha \vdash S(\alpha) \) and \( \alpha \vdash T(\alpha) \) be two types, with object interpretations \( S_0 \) and \( T_0 \) and relational interpretations \( S_1 \) and \( T_1 \). The interpretation of the product \( \alpha \vdash S(\alpha) \times T(\alpha) \) should be an appropriate product of interpretations; that is, the object interpretation should map a set \( A \) to \( S_0(A) \times T_0(A) \) and the relational interpretation should map a relation \( R \) to \( S_1(R) \times T_1(R) \), with the product of two relations defined in the obvious way. For the Identity Extension Lemma to hold, we need \( S_1(\text{Eq}(A)) \times T_1(\text{Eq}(A)) \) to be the same as \( \text{Eq}(S_0(A) \times T_0(A)) \). Here, the equality relation \( \text{Eq}(A) \) on a set \( A \) maps \( (a, b) : A \times A \) to the type \( \text{Id}(a, b) \) of proofs of equality between \( a \) and \( b \), so that \( a \) and \( b \) are related iff \( \text{Id}(a, b) \) is inhabited, *i.e.*, iff \( a \) is identical to \( b \). By the induction hypothesis, \( S_1(\text{Eq}(A)) \) is \( \text{Eq}(S_0(A)) \), and similarly for \( T \), so we need to show that \( \text{Eq}(S_0(A)) \times \text{Eq}(T_0(A)) \) is \( \text{Eq}(S_0(A) \times T_0(A)) \). But this is not necessarily the case since the identity type on a product is in general not identical to the product of identity types, but rather just suitably isomorphic. So the interpretation of \( \alpha \vdash S(\alpha) \times T(\alpha) \) is not necessarily an indexed or fibered functor (in the settings of [2] and [5], respectively).

Three ways to fix this problem come to mind. Firstly, we can attempt to change the meta-theory, by, *e.g.*, imposing an additional axiom asserting that two logically equivalent propositions are definitionally equal. We do not pursue this approach here: the goal of our framework is to *directly subsume the important models in their natural meta-theories*, as per criteria 1 and 2 above, rather than require the user to augment the meta-theory with *ad hoc* axioms to make the shoe fit. The second possibility is to use the syntactic analogue of strictification, pursued in, *e.g.*, [1]. The idea is that instead of interpreting a closed type as a set \( A \) (on the object level), we interpret it as a set \( A \) endowed with a relation \( R_A \) that is *isomorphic*, but not necessarily identical, to the canonical discrete relation \( \text{Eq}_A \). The chosen equality relation on the set \( A \) — more precisely, on the entire structure \( (A, (R_A, i : R_A \simeq \text{Eq}_A)) \) — will then be \( R_A \) rather
than $\text{Eq}_A$. This allows us to construct $R_A$ in a way that respects all type constructors on the nose, so that the aforementioned issue with $\text{Eq}(S_0(A)) \times \text{Eq}(T_0(A))$ not being identical to $\text{Eq}(S_0(A) \times T_0(A))$ is avoided. The problem, however, is that there can be many different ways to endow $A$ with a discrete relation $(R_A, i)$; in other words, the type of discrete relations on $A$ is not contractible. It is thus unclear whether and how this “discretized” version of Reynolds’ model is equivalent to the original, intended one.

Here we suggest a third approach: we record the isomorphisms witnessing the preservation of the Identity Extension Lemma for each type constructor, and propagate them through the construction. This means, however, that we can no longer interpret a type $\alpha \vdash T(\alpha)$ as a pair of maps $T_0 : \text{Set} \to \text{Set}$ and $T_1 : |\mathcal{R}| \to \mathcal{R}$; indeed, since the domain of $T_1$ is the discrete category $|\mathcal{R}|$, $T_1$ is not required to preserve isomorphisms in $\mathcal{R}$. As a result, even if we know that the pair $(T_0, T_1)$ satisfies the Identity Extension Lemma, its reindexing — defined by precomposition — might not. The upshot is that the obvious “$\lambda 2$-fibration” corresponding to Reynolds’ original model is not necessarily a fibration at all.

We solve this problem by specifying subcategories $\mathcal{M}(0) \subseteq \text{Set}$ and $\mathcal{M}(1) \subseteq \mathcal{R}$ of relevant isomorphisms that form a reflexive graph category with isomorphisms. Abstractly, this structure gives us two face maps (called $\partial_0$ and $\partial_1$ in [2]), which represent the domain and codomain projections, and a degeneracy (called $\text{Id}$ in [2]), which represents the equality functor. We interpret a type $\alpha \vdash T(\alpha)$ as a pair of functors $T_0 : \mathcal{M}(0) \to \mathcal{M}(0)$ and $T_1 : \mathcal{M}(1) \to \mathcal{M}(1)$ that together comprise a face map- and degeneracy-preserving reflexive graph functor, and interpret each term as a face map- and degeneracy-preserving reflexive graph natural transformation.

Since the domain of $T_1$ is $\mathcal{M}(1)$, $T_1$ preserves all relevant isomorphisms between relations, so the reindexing of $(T_0, T_1)$ is now well-defined. Choosing $\mathcal{M}(1)$ to contain the isomorphism between the two relations $\text{Eq}(S_0(A)) \times \text{Eq}(T_0(A))$ and $\text{Eq}(S_0(A) \times T_0(A))$ yields the satisfaction of the Identity Extension Lemma for products; other type constructors follow the same pattern. We note that although the preservation of isomorphisms on the relation level is sufficient to carry out the model construction, we formally require the preservation of relevant isomorphisms on the object level, too. This makes the framework more uniform and, moreover, leads to the novel notion of a categorical Reynolds’ model, in which interpretations of types are endowed with a functorial action on isomorphisms and all polymorphic functions respect this action. Furthermore, we go one level higher and use the ideas of Orsanigo [12] (and Ghani et al. [3], which it supersedes) to define a proof-relevant categorical Reynolds’ model, in which, additionally, witnesses of relatedness are themselves suitably related via a yet higher relation.

This “2-parametric” model of course does not arise as an instance of our framework since it requires additional structure — e.g., the concept of a 2-relation — pertaining to the higher notion of parametricity. Nevertheless, we would still like to be able to recognize it as a model parametric in the ordinary sense. Various definitions of parametricity for models of System F exist: [2, 5] are examples of “internal” approaches to parametricity, where a model is considered parametric if it is produced via a specified procedure that bakes in desired features of parametricity such as the Identity Extension Lemma. On the other hand, [4, 7, 10, 15] are examples of “external” approaches to
parametricity, in which reflexive graphs of models are used to endow models of interest with enough additional structure that they can reasonably be considered parametric. Surprisingly though, the proof-relevant model we give in Section 6 does not appear to satisfy any of these definitions, and in particular does not satisfy any of the external ones. The ability to construct a classifying reflexive graph seems to rely on an implicit assumption of proof-irrelevance, which we elaborate on in Section 6. However, we propose a new definition of a relationally parametric model of System F in Section 5 and show that it subsumes not only the two canonical parametric models of System F, but also the two novel ones we give in this paper. In particular, it subsumes the proof-relevant model given in Section 6.

The main contributions of this paper are as follows:

- We demonstrate that existing frameworks for the functorial semantics of relational parametricity for System F fail to directly subsume both canonical models of relational parametricity for System F.

- We solve this problem by developing a good abstract framework for relational parametricity that allows a choice of meta-theory, delivers both canonical relationally parametric models of System F as direct instances in a uniform way, and exposes properties that good models of System F parametricity should be expected to satisfy, e.g., guaranteeing that interpretations of terms, not just types, suitably commute with the degeneracy.

- We give a novel definition of a parametric model of System F, which is a hybrid of the external and internal approaches, and show that it subsumes both canonical models (expressed as instances of our framework).

- We give two novel relationally parametric models of System F — one of which is proof-relevant and can be seen as parametric in a higher sense (“2-parametric”) — and show that our definition recognizes both of these in a natural way, with the proof-irrelevant model arising as a direct instance of our framework.

Fibrational Preliminaries We give a brief introduction to fibrations, mainly to settle notation. More details can be found in, e.g., [7].

**Definition 1.** Let $U : \mathcal{E} \to \mathcal{B}$ be a functor. A morphism $g : Q \to P$ in $\mathcal{E}$ is cartesian over $f : X \to Y$ in $\mathcal{B}$ if $Ug = f$ and, for every $g' : Q' \to P$ in $\mathcal{E}$ with $Ug' = f \circ v$ for some $v : UQ' \to X$, there is a unique $h : Q' \to Q$ with $Uh = v$ and $g' = g \circ h$. A functor $U : \mathcal{E} \to \mathcal{B}$ is a fibration if, for every object $P$ of $\mathcal{E}$ and morphism $f : X \to UP$ of $\mathcal{B}$, there is a cartesian morphism in $\mathcal{E}$ with codomain $P$ over $f$.

If $U : \mathcal{E} \to \mathcal{B}$ is a fibration then $\mathcal{E}$ is its total category and $\mathcal{B}$ is its base category. An object $P$ in $\mathcal{E}$ is over its image $UP$, and similarly for morphisms. A morphism is vertical if it is over an identity morphism. We write $\mathcal{E}_X$ for the fiber over an object $X$ in $\mathcal{B}$, i.e., the subcategory of $\mathcal{E}$ of objects over $X$ and morphisms over $\text{id}_X$.

If $U : \mathcal{E} \to \mathcal{B}$ is a fibration, we call a cartesian morphism over $f$ with codomain $P$ a cartesian lifting of $f$ with codomain $P$ with respect to $U$. A cartesian lifting of $f$
with codomain $P$ with respect to $U$ need not be unique, but it is always unique up to vertical isomorphism. We are interested in fibrations in which representative cartesian liftings are specified, or chosen.

**Definition 2.** A fibration $U : E \to B$ is cloven if it comes with a choice of cartesian liftings, i.e., with one cartesian lifting of $f$ with codomain $P$ with respect to $U$ regarded as primary amongst all such cartesian liftings for each morphism $f$ in $B$ and object $P$ in $E$.

We emphasize that the choice of cartesian liftings is part of the structure that is given when a fibration is cloven. In this case one uses the phrase “the cartesian lifting” of $f$ with codomain $P$ to refer to the chosen such lifting, which we denote by $f_p^\uparrow$. Any time we consider categorical objects (e.g., categories, functors, etc.) with particular structures (e.g., products, adjoints, etc.) in this paper, we intend that those structures are chosen in this sense.

The function mapping each object $P$ of $E$ to the domain $f^*P$ of $f_p^\uparrow$ then extends to a functor $f^* : E_Y \to E_X$ mapping each morphism $k : P \to P'$ in $E_Y$ to the unique morphism $f^*k$ such that $k \circ f_p^\uparrow = f_p^\uparrow \circ f^*k$. The universal property of $f_p^\uparrow$ ensures the existence and uniqueness of $f^*k$. We call $f^*$ the substitution functor along $f$. We will be especially interested in cloven fibrations whose substitution functors are well-behaved:

**Definition 3.** A cloven fibration $U : E \to B$ is split if its substitution functors are such that $\text{id}^* = \text{id}$ and $(g \circ f)^* = f^* \circ g^*$.

We will later require even more structure of our split fibrations:

**Definition 4.** A split fibration $U : E \to B$ has a split generic object if there is an object $\Omega$ in $B$, together with a collection of isomorphisms $\theta_X$ mapping each morphism from $X$ to $\Omega$ in $B$ to an object of the fiber $E_X$ that is natural in $X$, i.e., is such that $\theta_Y(fg) = g^*(\theta_X(f))$ for every $f : X \to \Omega$ and $g : Y \to X$.

Seely [16] gave a sound categorical semantics of System F in $\lambda_2$-fibrations (presented as PL-categories). We will make good use of this result below.

## 2 Reflexive Graph Categories

Although Reynolds himself showed that his original approach to relational parametricity does not work in set theory, we can still use it as a guide for designing an abstract framework for parametricity. Instead of sets and relations, we consider abstract notions of “sets” and “relations”, and require them to be related as follows: i) for any relation $R$, there are two canonical ways of projecting an object out of $R$, corresponding to the domain and codomain operations, ii) for any object $A$, there is a canonical way of turning it into a relation, corresponding to the equality relation on $A$, and iii) if we start with an object $A$, turn it into a relation according to ii), and then project out an object according to i), we get $A$ back. This suggests that our abstract relations and the canonical operations on them can be organized into a reflexive graph structure: categories $\mathcal{X}_0$, $\mathcal{X}_1$, $\mathcal{X}_2$, $\cdots$. 

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\( \lambda_1 \) and functors \( f_T, f_L : \lambda_1 \to \lambda_0, d : \lambda_0 \to \lambda_0 \) such that \( f_T \circ d = d = f_L \circ d \), as is done in [2].

Since there are no set-theoretic models of System F ([14]), all of the reflexive graph structure identified above must to be internal to some ambient category \( \mathcal{C} \). In particular, \( \lambda_0 \) and \( \lambda_1 \) must be categories internal in \( \mathcal{C} \), and \( f_T, f_L, d \) must be functors internal in \( \mathcal{C} \). For Reynolds’ original model, the ambient category has types \( A : U_1 \) as objects and terms \( f : \Sigma_{A,B:U_1} A \to B \) as morphisms. Here, \( U_0 \) is the universe one level above the impredicative universe \( U_0 \); we will denote \( U_0 \) simply by \( U \) below. This ensures that \( U \) is an object in \( \mathcal{C} \). To model relations, we introduce:

\[
\text{isProp}(A) := \Pi_{a,b:A} \text{Id}(a,b) \\
\text{Prop} := \Sigma_{A:U} \text{isProp}(A)
\]

The type \( \text{Prop} \) of propositions singles out those types in \( U \) with the property that any two inhabitants, if they exist, are equal. Propositions can be used to model relations as follows: in Reynolds’ original model, \( a : A \) is related to \( b : B \) in at most one way under any relation \( R \) (either \( (a,b) \in R \) or not), so the type of proofs that \( (a,b) \in R \) is a proposition. Conversely, given \( R : A \times B \to \text{Prop} \), we consider \( a \) and \( b \) to be related by \( R \) iff \( R(a,b) \) is inhabited.

To see the universe \( U \) as a category \( \text{Set} \) internal to \( \mathcal{C} \) we take its object of objects \( \text{Set}_0 \) to be \( U \) and define its object of morphisms by \( \text{Set}_1 := \Sigma_{A,B:U} A \to B \). We define the category \( R \) of relations by giving its objects \( R_0 \) and \( R_1 \) of objects and morphisms, respectively:

\[
R_0 := \Sigma_{A,B:\text{Set}} A \times B \to \text{Prop} \\
R_1 := \Sigma_{((A_1,A_2),R_A),((B_1,B_2),R_B):R_0} \Sigma_{(f,g):(A_1 \to B_1) \times (A_2 \to B_2)} \Pi_{(a_1,a_2):(A_1 \times A_2)} R_A(a_1,a_2) \to R_B(f(a_1),g(a_2))
\]

We clearly have two internal functors from \( R \) to \( \text{Set} \) corresponding to the domain and codomain projections, respectively. We also have an internal functor \( \text{Eq} \) from \( \text{Set} \) to \( R \) that constructs an equality relation with \( \text{Eq} A := ((A,A), \text{Id}_A) \) and \( \text{Eq} ((A,B), f) := ((\text{Eq} A, \text{Eq} B), (f,f), \text{ap}_f) \). Here, the term \( \text{ap}_f : \text{Id}_A(a_1,a_2) \to \text{Id}_B(f(a_1), f(a_2)) \) is defined as usual by \( \text{Id} \)-induction and witnesses the fact that \( f \) respects equality.

These observations motivate the next two definitions, in which we denote the category of categories and functors internal to \( \mathcal{C} \) by \( \text{Cat}(\mathcal{C}) \), and assume \( \mathcal{C} \) is locally small and has all finite products. (A category is locally small if each of its hom-sets is small, \( i.e. \), is a set rather than a proper class.)

**Definition 5.** A reflexive graph structure \( \mathcal{X} \) on a category \( \mathcal{C} \) consists of:

- objects \( \mathcal{X}(0) \) and \( \mathcal{X}(1) \) of \( \mathcal{C} \)
- distinct arrows \( \mathcal{X}(f_+) : \mathcal{X}(1) \to \mathcal{X}(0) \) for \( * : \text{Bool} \)
- an arrow \( \mathcal{X}(d) : \mathcal{X}(0) \to \mathcal{X}(1) \)

such that \( \mathcal{X}(f_+) \circ \mathcal{X}(d) = \text{id} \).
The requirement that the two face maps \( \mathcal{X}(f_T) \) and \( \mathcal{X}(f_L) \) are distinct is to ensure that there are enough relations for the notion of relation-preservation to be meaningful. Otherwise, as also observed in [2], we could see any category \( C \) as supporting a trivial reflexive graph structure whose only relations are the equality ones. For readers familiar with [7], the condition \( \mathcal{X}(f_T) \neq \mathcal{X}(f_L) \) serves a purpose similar to that of the requirement in Definition 8.6.2 of [7] that the fiber category \( F_1 \) over the terminal object in \( C \) is the category of relations in the preorder fibration \( D \rightarrow E \) on the fiber category \( E_1 \) over the terminal object in \( B \). Both conditions imply that some relations must be heterogeneous. But while in [7] relations are obtained in a standard way as predicates (given by a preorder fibration) over a product, we do not assume that relations are constructed in any specific way, but rather only that the abstract operations on relations suitably interact. Moreover, since the two face maps \( \mathcal{X}(f_T) \) and \( \mathcal{X}(f_L) \) are distinct, any morphism generated by the face maps and the degeneracy \( \mathcal{X}(d) \) must be one of the seven distinct maps \( \text{id}_{\mathcal{X}(0)}, \text{id}_{\mathcal{X}(1)}, \mathcal{X}(f_k), \mathcal{X}(d), \text{and } \mathcal{X}(d) \circ \mathcal{X}(f_k) \) for \(* : \text{Bool} \). Every such morphism thus has a canonical representation.

**Definition 6.** A reflexive graph category (on \( C \)) is a reflexive graph structure on \( \text{Cat}(C) \).

**Example 7 (PER model).** We take the ambient category \( C \) to be the category of \( \omega \)-sets, given in Definition 6.3 of [9]. We construct a reflexive graph category, which we call \( \mathcal{R}_{\text{PER}} \), as follows. The internal category \( \mathcal{R}_{\text{PER}}(0) \) of “sets” is the category \( \mathcal{M}' \) given in Definition 8.4 of [9]. Informally, the objects of \( \mathcal{M}' \) are partial equivalence relations on \( N \), and the morphisms are realizable functions that respect such relations. To define the internal category \( \mathcal{R}_{\text{PER}}(1) \) of “relations”, we first construct its object of objects. The carrier of this \( \omega \)-set is the set of pairs of the form \( R := ((A_d, A_e), R_A) \), where \( A_d \) and \( A_e \) are partial equivalence relations and \( R_A \) is a saturated predicate on the product \( \text{PER} A_d \times A_e \). A saturated predicate on a \( \text{PER} \) is a predicate on \( N \) such that \( A_1 \sim_A A_2 \) and \( R(A_1) \) imply \( R(A_2) \). To finish the construction of our object of objects for \( \mathcal{R}_{\text{PER}}(1) \) we take any pair \( ((A_d, A_e), R_A) \) as above to be realized by any natural number.

The carrier of the object of morphisms for \( \mathcal{R}_{\text{PER}}(1) \) comprises all pairs of the form

\[
\left\{ \left( (A_d, A_e), R_A \right), (B_d, B_e), (R_B) \right\}, \left\{ m_1 \right\}_{A \rightarrow B_1}, \left\{ m_2 \right\}_{A \rightarrow B_2} \}
\]

satisfying the condition that, for any \( k, l \) such that \( k \sim_A k, l \sim_A l, \) and \( R_A((k, l)) \) holds, \( R_B((m_1 \cdot k, m_2 \cdot l)) \) holds as well. The first component records the domain and codomain of the morphism and the second component is a pair of equivalence classes under the specified exponential \( \text{PERs} \). As in [9], we denote the application of the \( n^\text{th} \) partial recursive function to a natural number \( a \) in its domain by \( n \cdot a \). To finish the construction of the object of morphisms for \( \mathcal{R}_{\text{PER}}(1) \), we take a pair of pairs as above to be realized by a natural number \( k \) iff \( \text{fst}(k) \sim_{A \rightarrow B_1} m_1 \) and \( \text{snd}(k) \sim_{A \rightarrow B_2} m_2 \).

We again have two internal functors \( \mathcal{R}_{\text{PER}}(f_T) \) and \( \mathcal{R}_{\text{PER}}(f_L) \) from \( \mathcal{R}_{\text{PER}}(1) \) to \( \mathcal{R}_{\text{PER}}(0) \) corresponding to the two projections. We also have an equality functor \( \text{Eq} \) from \( \mathcal{R}_{\text{PER}}(0) \) to \( \mathcal{R}_{\text{PER}}(1) \) whose action on objects is given by \( \text{Eq} A := ((A, A), \Delta_A) \), where \( \Delta_A(k) \) iff \( \text{fst}(k) \sim_A \text{snd}(k) \), and whose action on morphisms is given by

\[
\text{Eq} \left( (A, B), \left\{ m \right\}_{A \rightarrow B} \right) := \left( \left( \text{Eq} A, \text{Eq} B \right), \left\{ m \right\}_{A \rightarrow B} \right)
\]
Example 8 (Reynolds’ model). We obtain a reflexive graph category $\mathcal{R}_{\text{REY}}$ by taking $\mathcal{R}_{\text{REY}}(0) := \text{Set}$, $\mathcal{R}_{\text{REY}}(1) := \mathcal{R}$, and $\mathcal{R}_{\text{REY}}(d) := \text{Eq}$, and letting $\mathcal{R}_{\text{REY}}(f_T)$ and $\mathcal{R}_{\text{REY}}(f_L)$ be the functors corresponding to the domain and codomain projections, respectively.

If $\mathcal{X}$ is a reflexive graph category, then the discrete graph category $|\mathcal{X}|$ and the product reflexive graph category $\mathcal{X}^n$ for $n \in \mathbb{N}$ are defined in the obvious ways: $|\mathcal{X}(l)|$ has the same objects as $\mathcal{X}(l)$ but only the identity morphisms, and $(\mathcal{X} \times \mathcal{Y})(l) = \mathcal{X}(l) \times \mathcal{Y}(l)$ for $l \in \{0, 1\}$. For the latter, the product on the right-hand side is a product of internal categories, which exists because $\mathcal{C}$ has finite products by assumption.

If $\mathcal{C}$ is an internal category, we denote by $C_0$ and $C_1$ the objects of $\mathcal{C}$ representing the objects and morphisms of $\mathcal{C}$, respectively. If $F : \mathcal{C} \to \mathcal{D}$ is an internal functor, we denote by $F_0 : C_0 \to D_0$ and $F_1 : C_1 \to D_1$ the arrows of $\mathcal{D}$ representing the object and morphisms parts of $F$, respectively. Also:

Notation 9. We will use the following notation with respect to an internal category $\mathcal{C}$ in $\mathcal{C}$:

- Given a “generalized object” $a : J \to C_0$ (with $J$ arbitrary), we denote by $\text{id}_C[a]$ the arrow $\text{id}_C \circ a$, where $\text{id}_C : C_0 \to C_1$ is the arrow representing identity morphisms in $\mathcal{C}$.
- For a “generalized morphism” $f : J \to C_1$ (with $J$ arbitrary), we denote by $s_C[f]$ and $t_C[f]$ the arrows $s_C \circ f$ and $t_C \circ f$ respectively, where $s_C, t_C : C_1 \to C_0$ are the arrows representing the source and target operations in $\mathcal{C}$.
- For generalized morphisms $f, g : J \to C_1$ such that $t_C[f] = s_C[g]$, we denote by $g \circ_C f$ the arrow $\text{comp}_C \circ (f, g)$, where $\text{comp}_C : \text{pullback}(t_C, s_C) \to C_1$ is the arrow representing composition in $\mathcal{C}$, its domain $\text{pullback}(t_C, s_C)$ is the pullback of the two arrows $t_C$ and $s_C$, and $(f, g)$ is the canonical morphism into this pullback.
- We say that $f : J \to C_1$ is an isomorphism if there exists a $g : J \to C_1$ such that $s_C[f] = t_C[g], s_C[g] = t_C[f]$ and $f \circ_C g = \text{id}_C[s_C[g]], g \circ_C f = \text{id}_C[s_C[f]]$. If such a $g$ exists, it is necessarily unique and hence will be denoted by $f^{-1}$.

Given a reflexive graph category $\mathcal{X}$ axiomatizing the sets and relations, an obvious first attempt at pushing Reynolds’ original idea through is to take the interpretation $\llbracket T \rrbracket$ of a type $\pi \vdash T$ with $n$ free type variables to be a pair $([T](0), [T](1))$, where $[T](0) : |\mathcal{X}(0)|^n \to \mathcal{X}(0)$ and $[T](1) : |\mathcal{X}(1)|^n \to \mathcal{X}(1)$ are functions giving the “set” and “relation” interpretations of the type $T$. Although as explained in the introduction, this approach will need some tweaking — we will need to endow $[T](0)$ and $[T](1)$ with actions on some morphisms — it suggests:

Definition 10. Let $\mathcal{X}$ and $\mathcal{Y}$ be reflexive graph categories. A reflexive graph functor $\mathcal{F} : \mathcal{X} \to \mathcal{Y}$ is a pair $(\mathcal{F}(0), \mathcal{F}(1))$ of functors such that $\mathcal{F}(0) : \mathcal{X}(0) \to \mathcal{Y}(0)$ and $\mathcal{F}(1) : \mathcal{X}(1) \to \mathcal{Y}(1)$.
Writing \( T_0 \) for \( [T](0) \) and \( T_1 \) for \( [T](1) \), we recall from the introduction that \( T_0 \) and \( T_1 \) should be appropriately related via the domain and codomain projections and the equality functor. Since the two face maps \( \mathcal{X}(f_*) \) now model the projections, and the degeneracy \( \mathcal{X}(d) \) models the equality functor, we end up with the following conditions:

\[
\begin{align*}
\text{i) for each object } \overline{R} \text{ in } \mathcal{X}(1)^n, \text{ we have } \mathcal{X}(f_*) T_1(\overline{R}) &= T_0(\mathcal{X}(f_*)^n \overline{R}) \\
\text{ii) for each object } \overline{A} \text{ in } \mathcal{X}(0)^n, \text{ we have } \mathcal{X}(d) T_0(\overline{A}) &= T_1(\mathcal{X}(d)^n \overline{A})
\end{align*}
\]

We examine what these conditions imply for Reynolds’ model by considering the product hypothesis, \( S \) and \( T \) are interpreted as pairs \( (S_0, S_1) \) and \( (T_0, T_1) \), where \( S_0, T_0 : \text{Set}_0 \rightarrow \text{Set}_0 \) and \( S_1, T_1 : R_0 \rightarrow R_0 \) satisfy \( i \) and \( ii \). The interpretation of a product should be a product of interpretations, i.e., \( (S \times T)_0 A := S_0(A) \times T_0(A) \) and \( (S \times T)_1 R := S_1(R) \times T_1(R) \). It remains to be seen that this interpretation satisfies \( i \) and \( ii \). Fix a relation \( R \) on \( A \) and \( B \). Condition \( i \) entails that \( S_1(R) \) is \( ((S_0(A), S_0(B)), R_S) \) and \( T_1(R) := ((T_0(A), T_0(B)), R_T) \) for some \( R_S \) and \( R_T \). Thus \( S_1(R) \times T_1(R) \) has the form \( ((S_0(A) \times T_0(A), S_0(B) \times T_0(B)), R_{S \times T}) \), where \( R_{S \times T} \) maps a pair of pairs \( ((a, b), (c, d)) \) to \( R_S(a, c) \times R_T(b, d) \). Thus \( ii \) is satisfied simply by construction, which leads us to define:

**Definition 11.** A reflexive graph functor \( F : \mathcal{X} \rightarrow \mathcal{Y} \) is face map-preserving if the following diagram in \( \text{Cat}(C) \) commutes for all \( * \in \text{Bool} \):

\[
\begin{array}{ccc}
\mathcal{X}(1) & \xrightarrow{F(1)} & \mathcal{Y}(1) \\
\mathcal{X}(0) & \xrightarrow{F(0)} & \mathcal{Y}(0)
\end{array}
\]

In Reynolds’ model, condition \( ii \) gives that \( S_1(\text{Eq}(A)) \) is \( \text{Eq}(S_0(A)) \) for any set \( A \), and similarly for \( T \). We thus need to show that \( \text{Eq}(S_0(A)) \times \text{Eq}(T_0(A)) \) is \( \text{Eq}(S_0(A) \times T_0(A)) \). But while the domains and codomains of these two relations agree (all are \( S_0(A) \times T_0(A) \)), the former maps \( ((a, b), (c, d)) \) to \( \text{Id}(a, c) \times \text{Id}(b, d) \), while the latter maps it to \( \text{Id}((a, b), (c, d)) \). These two types are not necessarily identical, but they are isomorphic (i.e., there are functions back and forth that compose to identity on both sides).

We thus relax condition \( ii \) to allow an isomorphism \( \varepsilon_T(\overline{A}) : \mathcal{X}(d) T_0(\overline{A}) \cong T_1(\mathcal{X}(d)^n \overline{A}) \). In fact, we can require more: since the domains and codomains of \( \mathcal{X}(d) T_0(\overline{A}) \) and \( T_1(\mathcal{X}(d)^n \overline{A}) \) coincide by condition \( i \), we can insist that both projections map the isomorphism \( \varepsilon_T(\overline{A}) \) to the identity morphism on \( T_0(\overline{A}) \). This coherence condition is a natural counterpart to the equation \( \mathcal{X}(f_*) \circ \mathcal{X}(d) = \text{id} \), and turns out to be not just a design choice but a necessary requirement: in Reynolds’ model, for instance, the proof that the interpretations of \( \forall \)-types (as defined later) suitably commute with the functor \( \text{Eq} \) depends precisely on the morphisms underlying the maps \( \varepsilon_T(\overline{A}) \) being identities. This suggests:
The interpretation of a term $\eta$ and the isomorphism of relations from $S$ following diagram in $\text{Cat}(C)$ commutes up to a given natural isomorphism $\varepsilon_{\mathcal{F}}$ satisfying the coherence condition $\mathcal{Y}(f_1) \circ \varepsilon_{\mathcal{F}} = \text{id}_{\mathcal{Y}(0)}[\mathcal{F}(0)_{\alpha}]$ for $\alpha \in \text{Bool}$:

\[
\begin{array}{c}
\mathcal{X}(0) \xrightarrow{\mathcal{F}(0)} \mathcal{Y}(0) \\
\mathcal{X}(d) \xrightarrow{\mathcal{F}(1)} \mathcal{Y}(d) \\
\mathcal{X}(1) \xrightarrow{\mathcal{F}(1)} \mathcal{Y}(1)
\end{array}
\]

As a first approximation, we can try to interpret a type $\tau \vdash T$ with $n$ free type variables as a face map- and degeneracy-preserving reflexive graph functor $(T_0, T_1) : |\mathcal{X}|^n \to \mathcal{X}$. Reynolds’ original idea for interpreting terms suggests that the interpretation of a term $\tau; x : S \vdash t : T$ should be a (vacuously) natural transformation $t_0 : S_0 \to T_0$. As observed in [5], the Abstraction Theorem can then be formulated as follows: there is a (vacuously) natural transformation $t_1 : S_1 \to T_1$ such that, for any object $\bar{R}$ in $\mathcal{X}(1)^n$, we have $\mathcal{X}(f_1) t_1(\bar{R}) = t_0(\mathcal{X}(f_1)^n \bar{R})$. To see that this does indeed give what we want, we revisit Reynolds’ model. There, the face maps are the domain and codomain projections and an object $R$ in $\mathcal{X}(1)^n$ is an $n$-tuple of relations. Denote $\mathcal{X}(f_1)^n \bar{R}$ by $\mathcal{A}$ and $\mathcal{X}(f_1)^n \bar{B}$ by $\mathcal{B}$. Then $t_1(\bar{R})$ is a morphism of relations from $S_1(\bar{R})$ to $T_1(\bar{R})$ and, since $S_1$ and $T_1$ are face map-preserving, $S_1(\bar{R}) := ((S_0(\mathcal{A}), S_0(\mathcal{B})), R_S)$ and $T_1(\bar{R}) := ((T_0(\mathcal{A}), T_0(\mathcal{B})), R_T)$ for some $R_S$ and $R_T$. By definition, $t_1(\bar{R})$ gives maps $f : S_0(\mathcal{A}) \to T_0(\mathcal{A})$, $g : S_0(\mathcal{B}) \to T_0(\mathcal{B})$, together with a map $h : \Pi_{(a_1, a_2) : S_0(\mathcal{A}) \times S_0(\mathcal{B})} R_S(a_1, a_2) \to R_T(f(a_1), g(a_2))$ stating precisely that $f$ and $g$ related inputs to related outputs. By definition, $\mathcal{X}(f_1) t_1(\bar{R})$ is $((S_0(\mathcal{A}), T_0(\mathcal{A})), f)$ and $\mathcal{X}(f_1) t_1(\bar{R})$ is $((S_0(\mathcal{B}), T_0(\mathcal{B})), g)$, so the condition that $\mathcal{X}(f_1) t_1(\bar{R})$ is $t_0(\mathcal{X}(f_1)^n \bar{R})$ implies that the maps underlying $t_0(\mathcal{A})$ and $t_0(\mathcal{B})$ must be $f$ and $g$, respectively, and so must indeed map related inputs to related outputs, as witnessed by $h$. Pairing the natural transformations $t_0$ and $t_1$ motivates:

**Definition 12.** Let $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{Y}$ be reflexive graph functors. A reflexive graph natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ is a pair $(\eta(0), \eta(1))$ of natural transformations $\eta(0) : \mathcal{F}(0) \to \mathcal{G}(0)$ and $\eta(1) : \mathcal{F}(1) \to \mathcal{G}(1)$.

The Abstraction Theorem then further suggests defining:

**Definition 13.** Let $\mathcal{F}, \mathcal{G} : \mathcal{X} \to \mathcal{Y}$ be reflexive graph functors. A reflexive graph natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ between two face map-preserving reflexive graph functors is face map-preserving if for any $\ast \in \text{Bool}$ we have

$$\mathcal{Y}(f_1) \circ \eta(1) = \eta(0) \circ \mathcal{X}(f_0)$$

The interpretation of a term $\tau; x : S \vdash t : T$ should then be a face map-preserving natural transformation from $(S_0, S_1)$ to $(T_0, T_1)$. We also have the dual notion:

**Definition 14.** A reflexive graph natural transformation $\eta : \mathcal{F} \to \mathcal{G}$ between two degeneracy-preserving reflexive graph functors $(\mathcal{F}, \varepsilon_{\mathcal{F}})$ and $(\mathcal{G}, \varepsilon_{\mathcal{G}})$ is degeneracy-preserving if for any $\ast \in \text{Bool}$, we have

$$(\eta(1) \circ \mathcal{X}(d)) \circ \varepsilon_{\mathcal{F}} = \varepsilon_{\mathcal{G}} \circ \mathcal{Y}(\mathcal{X}(d) \circ \eta(0))$$
Intuitively, the above equation represents the commutativity of the following diagram in the internal category \( \mathcal{Y}(1) \):

\[
\begin{array}{ccc}
\mathcal{Y}(d)_0 \circ F(0) & \overset{\varepsilon_F}{\longrightarrow} & F(1)_0 \circ \mathcal{X}(d)_0 \\
\mathcal{Y}(d)_1 \circ \eta(0) & \quad & \eta(1) \circ \mathcal{X}(d)_0 \\
\mathcal{Y}(d)_0 \circ G(0) & \overset{\varepsilon_G}{\longrightarrow} & G(1)_0 \circ \mathcal{X}(d)_0
\end{array}
\]

There is no explicit analogue of Definition 15 in Reynolds’ model for the following reason: Reynolds’ model (as well as the PER model) is proof-irrelevant, in the precise sense that the functor \( \langle \mathcal{X}(f_L), \mathcal{X}(f_T) \rangle \) is faithful, and this condition is sufficient to guarantee that any face map-preserving natural transformation is automatically degeneracy-preserving as well. This may or may not be the case in proof-relevant models (although in the model from Section 6 it is), so we explicitly restrict attention below only to those natural transformations that are face map- and degeneracy-preserving (as also done in [2]), and omit further mention of these properties.

We have the usual laws of identity and composition of reflexive graph functors and natural transformations:

**Definition 16.** Given a reflexive graph category \( \mathcal{X} \), the identity reflexive graph functor \( 1_{\mathcal{X}} : \mathcal{X} \to \mathcal{X} \) is defined as follows:

- \( 1_{\mathcal{X}}(l) \) is the identity functor on \( \mathcal{X}(l) \)
- \( \varepsilon_{1_{\mathcal{X}}} := \text{id}_{\mathcal{X}(1)}[\mathcal{X}(d)_0] \)

**Definition 17.** Given two reflexive graph functors \( F : \mathcal{X} \to \mathcal{Y} \) and \( G : \mathcal{Y} \to \mathcal{Z} \), let \( G \circ F : \mathcal{X} \to \mathcal{Z} \) be the reflexive graph functor defined as follows:

- \((G \circ F)(l) := G(l) \circ F(l)\)
- \(\varepsilon_{G \circ F} := (G(1)_0 \circ \varepsilon_F) \circ (\varepsilon_G \circ F(0)_0)\)

**Definition 18.** Given a reflexive graph functor \( F : \mathcal{X} \to \mathcal{Y} \), the identity reflexive graph natural transformation \( 1_F : F \to F \) is defined by \( 1_F(l) = \text{id}_{\mathcal{Y}(l)}[F(l)_0] \).

**Definition 19.** Given reflexive graph functors \( F, G, H : \mathcal{X} \to \mathcal{Y} \) and reflexive graph natural transformations \( \eta_1 : F \to G \) and \( \eta_2 : G \to H \), let \( \eta_2 \circ \eta_1 : F \to H \) be the reflexive graph natural transformation defined by \( (\eta_2 \circ \eta_1)(l) := \eta_2(l) \circ \eta_1(l) \).

**Definition 20.** Given reflexive graph functors \( F : \mathcal{X} \to \mathcal{Y} \) and \( G_1, G_2 : \mathcal{Y} \to \mathcal{Z} \), and a reflexive graph natural transformation \( \eta : G_1 \to G_2 \), let \( \eta \circ F : G_1 \circ F \to G_2 \circ F \) be the reflexive graph natural transformation defined by \( (\eta \circ F)(l) := \eta(l) \circ F(l)_0 \).

**Definition 21.** Given reflexive graph functors \( F_1, F_2 : \mathcal{X} \to \mathcal{Y} \) and \( G : \mathcal{Y} \to \mathcal{Z} \), and a reflexive graph natural transformation \( \eta : F_1 \to F_2 \), let \( \eta \circ G : G \circ F_1 \to G \circ F_2 \) be the reflexive graph natural transformation defined by \( (\eta \circ G)(l) := G(l)_1 \circ \eta(l) \).
One basic example of a reflexive graph functor which will be used often and will end up interpreting type variables is the projection:

**Definition 22.** Given a reflexive graph category $\mathcal{X}$ and $0 \leq i < n$, the “$i$-th projection” reflexive graph functor $\text{pr}_n^i : \mathcal{X}^n \to \mathcal{X}$ is defined as follows:

- $\text{pr}_n^i(l)$ is the internal functor projecting out the $i$-th component
- $\varepsilon_{\text{pr}_n^i} := \text{id}_{\mathcal{X}(1)} [\mathcal{X}(d)_0 \circ \text{pr}_n^i(0)]$

Dually, we have the following:

**Definition 23.** Given reflexive graph functors $F_0, \ldots, F_{m-1} : \mathcal{X} \to \mathcal{Y}$, let $\langle F_0, \ldots, F_{m-1} \rangle$ be the reflexive graph functor from $\mathcal{X}$ to $\mathcal{Y}^m$ defined as follows:

- $\langle F_0, \ldots, F_{m-1} \rangle(l) := \langle F_0(l), \ldots, F_{m-1}(l) \rangle$
- $\varepsilon_{\langle F_0, \ldots, F_{m-1} \rangle} := \langle \varepsilon_{F_0}, \ldots, \varepsilon_{F_{m-1}} \rangle$

**Definition 24.** Given reflexive graph functors $F_0, \ldots, F_{m-1}, G_0, \ldots, G_{m-1} : \mathcal{X} \to \mathcal{Y}$, and reflexive graph natural transformations $\eta_0 : F_0 \to G_0, \ldots, \eta_{m-1} : F_{m-1} \to G_{m-1}$, let $\langle \eta_0, \ldots, \eta_{m-1} \rangle : \langle F_0, \ldots, F_{m-1} \rangle \to \langle G_0, \ldots, G_{m-1} \rangle$ be the reflexive graph natural transformation defined by $\langle \eta_0, \ldots, \eta_{m-1} \rangle(l) := \langle \eta_0(l), \ldots, \eta_{m-1}(l) \rangle$.

**Lemma 25.** We have the following properties:

1. The identity reflexive graph functor serves as the identity for the composition of reflexive graph functors.
2. The composition of reflexive graph functors is associative.
3. The identity reflexive graph natural transformation serves as the identity for the composition of reflexive graph natural transformations.
4. The composition of reflexive graph natural transformations is associative.
5. The composition $(\cdot) \circ F$ of a reflexive graph functor and a reflexive graph natural transformation is functorial.
6. The composition $G \circ (\cdot)$ of a reflexive graph natural transformation and a reflexive graph functor is functorial.

### 3 Reflexive Graph Categories with Isomorphisms

As noted above, if we try to interpret a type $\bar{\alpha} \vdash T$ as a reflexive graph functor $[T] : \mathcal{X}^n \to \mathcal{X}$ we encounter a problem with contravariance. Specifically, if $\alpha \vdash A$ and $\alpha \vdash B$ are types, then to interpret the function type $\alpha \vdash A \to B$ as the exponential of $[A]$ and $[B]$, $[A \to B](0)$ must map each object $X$ to the exponential $([A](0) \to X)$ and each morphism $f : X \to Y$ to a morphism from $([A](0) \to X)$ to $([B](0) \to Y)$. But there is no canonical way to construct
a morphism of this type because $[A](0) \ f$ goes in the wrong direction. This is a well-known problem that is unrelated to parametricity.

The usual solution is to require the domains of the functors interpreting types to be discrete, so that $[T] : |\mathcal{X}|^n \to \mathcal{X}$. However, as noted in the introduction, this will not work in our setting. Consider types $\alpha \vdash S(\alpha)$ and $\cdot \vdash T$. By the induction hypothesis, $[S] : |\mathcal{X}| \to \mathcal{X}$ and $[T] : 1 \to \mathcal{X}$ are face map- and degeneracy-preserving reflexive graph functors. The interpretation of the type $\cdot \vdash S[\alpha := T]$ should be given by the composition $[S] \circ [T] : 1 \to \mathcal{X}$, which should be a face map- and degeneracy-preserving functor. While preservation of face maps is easy to prove, preservation of degeneracies poses a problem: writing $S_0$ and $S_1$ for $[S](0)$ and $[S](1)$, and similarly for $T$, we need $S_1(T_1)$ to be isomorphic to the degeneracy $d(S_0(T_0))$. By assumption, $T_1$ is isomorphic to the degeneracy $d(T_0)$, and $S_1(d(T_0))$ is isomorphic to $d(S_0(T_0))$, so if we knew that $S_1$ mapped isomorphic relations to isomorphic relations we would be done. But since the domain of $S_1$ is $|\mathcal{X}|(1)$, there is no reason that it should preserve non-identity isomorphisms of $\mathcal{X}(1)$.

In this paper we solve this contravariance problem in a different way. We first note that the issue does not arise if $[A](0) \ f$ is an isomorphism, even if that isomorphism is not the identity. This leads us to require, for each $l \in \{0, 1\}$, a wide subcategory $\mathcal{M}(l) \subseteq \mathcal{X}(l)$ such that every morphism in $\mathcal{M}(l)$ is in fact an isomorphism.

**Definition 26.** Given a reflexive graph category $\mathcal{X}$, a reflexive graph subcategory of $\mathcal{X}$ is a reflexive graph category $\mathcal{M}$ together with a reflexive graph “inclusion” functor $\mathcal{I} : \mathcal{M} \to \mathcal{X}$ such that

- $\mathcal{I}(l)_0$ and $\mathcal{I}(l)_1$ are monic for $l \in \{0, 1\}$
- $\mathcal{I}(0) \circ \mathcal{M}(f_\ast) = \mathcal{X}(f_\ast) \circ \mathcal{I}(1)$ for $\ast \in \text{Bool}$
- $\mathcal{I}(1) \circ \mathcal{M}(d) = \mathcal{X}(d) \circ \mathcal{I}(1)$

The subcategory $(\mathcal{M}, \mathcal{I})$ is wide if $\mathcal{I}(l)_0$ is an isomorphism for $l \in \{0, 1\}$.

The last two conditions in Definition 26 guarantee that $\mathcal{I}$ preserves face maps and degeneracies on the nose. To simplify the presentation, we treat $\mathcal{M}(l)$ as a subcategory of $\mathcal{X}(l)$ and avoid explicit mentions of $\mathcal{I}$ unless otherwise indicated.

**Definition 27.** A reflexive graph category with isomorphisms is a reflexive graph category $\mathcal{X}$ together with a wide reflexive graph subcategory $(\mathcal{M}, \mathcal{I})$ such that every morphism in $\mathcal{M}(l)$, $l \in \{0, 1\}$, is an isomorphism.

We view $\mathcal{M}(l)$ as selecting the relevant isomorphisms of $\mathcal{X}(l)$, in the sense that a morphism of $\mathcal{X}(l)$ is relevant iff it lies in the image of $\mathcal{I}(l)$. Given a reflexive graph category with isomorphisms $(\mathcal{X}, (\mathcal{M}, \mathcal{I}))$ we can now interpret a type $\exists \alpha \vdash T$ with $n$ free type variables as a reflexive graph functor $[T] : \mathcal{M}^n \to \mathcal{M}$. It is important that $[T]$ carries (tuples of) relevant isomorphisms to relevant isomorphisms: if $[T]$ were instead a functor from $\mathcal{M}^n$ to $\mathcal{X}$, then it would not be possible to define substitution (see Definition 32).

A trivial choice is to take $\mathcal{M} := |\mathcal{X}|$. Then $[T] : |\mathcal{X}|^n \to |\mathcal{X}|$ and $\varepsilon_{[T]}$ is necessarily the identity natural transformation, so $[T]$ preserves degeneracies on the nose.
This instantiation shows that, despite being motivated by Reynolds’ model, for which the Identity Extension Lemma holds only up to isomorphism, our framework can also uniformly subsume strict models of parametricity, for which the Identity Extension Lemma holds on the nose.

**Example 28** (PER model, continued). We take $\mathcal{M} := |\mathcal{R}_{\text{PER}}|.$

**Example 29** (A categorical version of Reynolds’ model). For each $l$, we take the objects of $\mathcal{M}(l)$ to be the objects of $\mathcal{R}_{\text{REY}}(l)$, and the morphisms of $\mathcal{M}(l)$ to be all isomorphisms of $\mathcal{R}_{\text{REY}}(l).$ For example, the morphisms of $\mathcal{M}(0)$ are

$$\{(i, j) : \text{Set}_1 \times \text{Set}_1 &
i i_d = j_c \times i_c = j_d \times j \circ i = \text{id} \times i \circ j = \text{id}\}$$

Here and at several places below we write $a = b$ for $\text{Id}(a, b)$ and $\{x : A \& B(x)\}$ for $\Sigma_{x:A} B(x)$ to enhance readability. Moreover, $\circ$ and $\text{id}$ are composition and identity in the category $\text{Set}$, and we use the subscripts $(\cdot)_d$ and $(\cdot)_c$ to denote the domain and codomain of a morphism. The first (or second) projection gives the required mono from $\mathcal{M}(0)$ to $\text{Set}_1.$ We denote the resulting reflexive graph category with isomorphisms by $\mathcal{R}_{\text{CREY}}.$

**Example 30** (Reynolds’ model, continued). As mentioned in the introduction, to push the constructions through it is sufficient to require preservation of isomorphisms on the relation level only. This means that on the set level, we can take the relevant isomorphisms to be just the identities, i.e., $\mathcal{M}(0) := |\mathcal{R}_{\text{REY}}(0)|.$ On the relation level, we take the objects of $\mathcal{M}(1)$ to be the objects of $\mathcal{R}_{\text{REY}}(1)$ — i.e., we take all relations — and the morphisms of $\mathcal{M}(1)$ to be those isomorphisms of $\mathcal{R}_{\text{REY}}(1)$ whose images under the two face maps are identities (this last condition is necessary since face maps must preserve relevant isomorphisms). Specifically, the morphisms of $\mathcal{M}(1)$ are

$$\{(i, j) : \mathcal{R}_1 \times \mathcal{R}_1 &
i i_d = j_c \times i_c = j_d \times j \circ i = \text{id} \times i \circ j = \text{id} \times i \top = \text{id} \times i \bot = \text{id}\}$$

Here, we use the subscripts $(\cdot)_\top$ and $(\cdot)_\bot$ to denote the image of a morphism in $\mathcal{R}_1$ under the corresponding face map.

With this infrastructure in place we can now interpret a term $\overline{x} : S \vdash t : T$ as a natural transformation from $\mathcal{I} \circ [S]$ to $\mathcal{I} \circ [T].$ Importantly, the components of such a natural transformation are drawn from $\mathcal{A}(l)$ (as witnessed by post-composition with $\mathcal{I}$), rather than just $\mathcal{M}(l),$ as would be the case if we interpreted $t$ as a natural transformation from $[S]$ to $[T].$ In fact, this latter interpretation would not even be sensible, since not every term gives rise to an isomorphism (most do not).

## 4 Cartesian Closed Reflexive Graph Categories With Isomorphisms

We want to interpret a type context of length $n$ as the natural number $n,$ types with $n$ free type variables as reflexive graph functors from $\mathcal{M}^n$ to $\mathcal{M},$ and terms with $n$
free type variables as natural transformations between reflexive graph functors with codomain \( \mathcal{X} \). Following the standard procedure, we first define, for each \( n \), a category \( \mathcal{M}^n \rightarrow \mathcal{M} \) to interpret expressions with \( n \) free type variables, and then combine these categories using the usual Grothendieck construction. This gives a fibration whose fiber over \( n \) is \( \mathcal{M}^n \rightarrow \mathcal{M} \).

**Definition 31.** The category \( \mathcal{M}^n \rightarrow \mathcal{M} \) is defined as follows:

- the objects are face map- and degeneracy-preserving reflexive graph functors from \( \mathcal{M}^n \) to \( \mathcal{M} \)
- the morphisms from \( \mathcal{F} \) to \( \mathcal{G} \) are the face map- and degeneracy-preserving reflexive graph natural transformations from \( \mathcal{I} \circ \mathcal{F} \) to \( \mathcal{I} \circ \mathcal{G} \)

If \( \mathcal{F} \) and \( \mathcal{G} \) are degeneracy-preserving then \( \mathcal{I} \circ \mathcal{F} \) and \( \mathcal{I} \circ \mathcal{G} \) are as well, and it is therefore sensible to require natural transformations between the latter two to be degeneracy-preserving. To move between the fibers we need a notion of substitution:

**Definition 32.** For any \( m \)-tuple \( \mathbf{F} := (\mathcal{F}_0, \ldots, \mathcal{F}_{m-1}) \) of objects in \( \mathcal{M}^n \rightarrow \mathcal{M} \), the functor \( \mathbf{F}^* \) from \( \mathcal{M}^m \rightarrow \mathcal{M} \) to \( \mathcal{M}^n \rightarrow \mathcal{M} \) is defined by \( \mathbf{F}^*(\mathcal{G}) := \mathcal{G} \circ (\mathcal{F}_0, \ldots, \mathcal{F}_{m-1}) \) for objects and \( \mathbf{F}^*(\eta) := \eta \circ (\mathcal{F}_0, \ldots, \mathcal{F}_{m-1}) \) for morphisms.

When giving a categorical interpretation of System F, a category for interpreting type contexts is also required. Writing \( \mathcal{R} \) for the tuple \((\mathcal{X}, (\mathcal{M}, \mathcal{I}))\), we define:

**Definition 33.** The category of contexts \( \text{Ctx}(\mathcal{R}) \) is given by:

- objects are natural numbers
- morphisms from \( n \) to \( m \) are \( m \)-tuples of objects in \( \mathcal{M}^n \rightarrow \mathcal{M} \)
- the identity \( \text{id}_n : n \rightarrow n \) has as its \( i \)th component the \( i \)th projection functor \( \text{pr}_i^n \)
- given morphisms \( \mathbf{F} : n \rightarrow m \) and \( \mathbf{G} = (\mathcal{G}_0, \ldots, \mathcal{G}_{k-1}) : m \rightarrow k \), the \( i \)th component of the composition \( \mathbf{G} \circ \mathbf{F} : n \rightarrow k \) is \( \mathbf{F}^*(\mathcal{G}_i) \)

That this is indeed a category follows from the lemma below:

**Lemma 34.** We have the following:

i) For any morphism \( \mathbf{F} = (\mathcal{F}_0, \ldots, \mathcal{F}_{m-1}) : n \rightarrow m \) in \( \text{Ctx}(\mathcal{R}) \) and \( 0 \leq i < m \), we have \( \mathbf{F}^*(\text{pr}_i^n) = \mathcal{F}_i \).

ii) For any natural number \( n \), \( (1_n)^* \) is the identity functor on \( |\mathcal{R}|^n \rightarrow \mathcal{R} \).

iii) For morphisms \( \mathbf{F} : n \rightarrow m \), \( \mathbf{G} : m \rightarrow k \) in \( \text{Ctx}(\mathcal{R}) \), we have \( (\mathbf{G} \circ \mathbf{F})^* = \mathbf{F}^* \circ \mathbf{G}^* \).

**Proof.** Parts i), ii) are easy to show. For part iii) let \( \mathbf{F} = (\mathcal{F}_0, \ldots, \mathcal{F}_{m-1}) \) and \( \mathbf{G} = (\mathcal{G}_0, \ldots, \mathcal{G}_{k-1}) \). Fix an object \( \mathcal{H} \) in \( \mathcal{M}^k \rightarrow \mathcal{M} \). The first component of \( (\mathbf{G} \circ \mathbf{F})^* (\mathcal{H}) \) is the reflexive graph functor whose component at level \( l \) is

\[
\mathcal{H}(l) \circ \left( \mathcal{G}_0(l) \circ (\mathcal{F}_0(l), \ldots, \mathcal{F}_{m-1}(l)), \ldots, \mathcal{G}_{k-1}(l) \circ (\mathcal{F}_0(l), \ldots, \mathcal{F}_{m-1}(l)) \right)
\]
On the other hand, the first component of $F^*(G^*(\mathcal{H}))$ is a reflexive graph functor whose component at level $l$ is

$$\mathcal{H}(l) \circ \langle G_0(l), \ldots, G_{k-1}(l) \rangle \circ \langle F_0(l), \ldots, F_{m-1}(l) \rangle$$

which is clearly equal to the above. The second component of $(G \circ F)^*(\mathcal{H})$ is the morphism

$$\left(\mathcal{H}(1) \circ \langle G_0(1), \ldots, G_{k-1}(1) \rangle \circ \langle F_0(l), \ldots, F_{m-1}(l) \rangle\right) \circ M(1)$$

We now have the chain of equalities in Figure 1, where the first equality follows by definition of $\circ M(1)^m$; the second one follows by functoriality of $\mathcal{H}(1)$; and the third one follows since $\circ M(1)$ commutes with precomposition in the ambient category. We use the same color to denote rewriting of equal subexpressions. This finishes the proof that $(G \circ F)^*$ and $F^* \circ G^*$ agree on objects. The proof that they agree on morphisms is easy.


Lemma 35. The category $\text{Ctx}(\mathcal{R})$ has a terminal object $0$ and products $(-) \times 1$.

Proof. The product of $n$ and 1 is $n + 1$; the first projection has as its $i$-th component the “$i$-th projection functor” $\text{pr}_i^{n+1}$ and the second projection has as its sole component the “$n$-th projection functor” $\text{pr}_{n+1}^{n+1}$.

The categories $\text{Ctx}(\mathcal{R})$ and $M^n \to \mathcal{M}$ can be combined to give:

Definition 36. The category $\int_n M^n \to \mathcal{M}$ is defined as follows:

- objects are pairs $(n, F)$, where $F$ is an object in $M^n \to \mathcal{M}$
- morphisms from $(n, F)$ to $(m, G)$ are pairs $(F, \eta)$, where $F : n \to m$ is a morphism in $\text{Ctx}(\mathcal{X})$ and $\eta : F \to F^*(G)$ is a morphism in $M^n \to \mathcal{M}$
The forgetful functor from Lemma 37.

Given any morphism $F: n \to m$ in $\mathsf{Ctx}(\mathcal{R})$ and an object $G$ in $\mathcal{M}^m \to \mathcal{M}$, the cartesian lifting of $F$ with respect to $G$ is defined to be the morphism $(F, \text{id}_{F^*(G)}): (n, F^*(G)) \to (m, G)$ in the total category $\int_n \mathcal{M}^n \to \mathcal{M}$. The induced reindexing functor is precisely $F^*$. \qed
To appropriately interpret arrow types we need the category $\mathcal{M}^n \to \mathcal{M}$ to be cartesian closed. For this we require more structure on the underlying reflexive graph category with isomorphisms. We define:

**Definition 38.** An internal category $C$ in $\mathcal{C}$ has a terminal object if it comes equipped with an arrow $1_C : 1 \to C_0$ with the following universal property:

- for any object $a : J \to C_0$ (with $J$ arbitrary), there is a unique morphism $!_C(a) : J \to C_1$ such that

  \[
  s_C[!_C(a)] = a \\
  t_C[!_C(a)] = 1_C \circ !J
  \]

It is possible to show that the above definition is equivalent to the standard one given e.g., in Section 7.2 of [7]. However, the explicit version will be more useful for us.

**Definition 39.** A reflexive graph category $\mathcal{X}$ has terminal objects if for each $l \in \{0, 1\}$ the category $\mathcal{X}(l)$ has a terminal object. The terminal objects are stable under face maps if for any $\ast \in \mathbb{Bool}$, the canonical morphism witnessing the commutativity of the diagram below is the identity:

\[
\begin{array}{ccc}
1 & \xrightarrow{1_{\mathcal{X}(1)}} & \mathcal{X}(1)_0 \\
\downarrow & & \downarrow \\
1_{\mathcal{X}(0)} & \xrightarrow{1_{\mathcal{X}(0)}} & \mathcal{X}(0)_0
\end{array}
\]

The terminal objects are stable under degeneracies if the canonical morphism $\eta^1_{\mathcal{X}}$ witnessing the commutativity of the diagram below is an isomorphism:

\[
\begin{array}{ccc}
1 & \xrightarrow{1_{\mathcal{X}(0)}} & \mathcal{X}(0)_0 \\
\downarrow & & \downarrow \\
1_{\mathcal{X}(1)} & \xrightarrow{1_{\mathcal{X}(1)}} & \mathcal{X}(1)_0
\end{array}
\]

**Definition 40.** A reflexive graph category $(\mathcal{X}, (\mathcal{M}, I))$ with isomorphisms has terminal objects if $\mathcal{X}$ has terminal objects. The terminal objects are stable under face maps if the terminal objects in $\mathcal{X}$ are stable under face maps. The terminal objects are stable under degeneracies if the terminal objects in $\mathcal{X}$ are stable under degeneracies and the (iso)morphism $\eta^1_{\mathcal{X}}$ is in the image of $I(1)$.

**Lemma 41.** If a reflexive graph category $(\mathcal{X}, (\mathcal{M}, I))$ with isomorphisms has terminal objects stable under face maps and degeneracies, then for each $n$, the category $\mathcal{M}^n \to \mathcal{M}$ has a terminal object.
Proof. We define the terminal object in $\mathcal{M}^n \to \mathcal{M}$ to be $1_n$, where

- $1_n(0) := 1_{X(l)} \circ !(\mathcal{M}(l)_n^0)$
- $1_n(1) := \text{id}_{\mathcal{M}(l)}[1_{X(l)}] \circ !(\mathcal{M}(l)_n^1)$
- $\varepsilon_n := \eta^1_{X(l)} \circ !(\mathcal{M}(0)_n^0)$

To show that $1_n$ is indeed a terminal object, take another object $\mathcal{F}$. The universal morphism from $\mathcal{F}$ into our candidate terminal object is the reflexive graph natural transformation whose component at level $l$ is $1_{X(l)}(\mathcal{F}(l)_0)$. To prove naturality, we need to show that

$$(1_{X(l)}(\mathcal{F}(l)_0) \circ t_{\mathcal{M}(l)_n^0}) \circ 1_{X(l)} \mathcal{F}(l)_1 = 1_n(1) \circ 1_{X(l)} (1_{X(l)}(\mathcal{F}(l)_0) \circ s_{\mathcal{M}(l)_n^0})$$

The target of both sides is $1_{X(l)} \circ !(\mathcal{M}(l)_n^0)$ so the equality follows from the universal property of $1_{X(l)}$. To prove that the candidate universal morphism is degeneracy-preserving, we need to show that

$$(1_{X(l)}(\mathcal{F}(1)_0) \circ \mathcal{M}(d)_n^0) \circ 1_{X(l)} \varepsilon_\mathcal{F} = \varepsilon_n \circ 1_{X(l)} (\mathcal{M}(d)_1 \circ 1_{X(0)}(\mathcal{F}(0)_0))$$

The target of both sides is $1_{X(l)} \circ !(\mathcal{M}(0)_n^0)$ so the equality again follows from the universal property of $1_{X(l)}$. The preservation of face maps follows by the exact same argument. This shows that our candidate universal morphism is indeed a proper morphism. Its uniqueness is obvious, once again by the universal property of $1_{X(l)}$. \[\square\]

**Definition 42.** An internal category $C$ in $C$ has products if it comes equipped with arrows $\times_C : C_0 \times C_0 \to C_0$ and $\text{fst}_C, \text{snd}_C : C_0 \times C_0 \to C_1$ such that

- $s_C[\text{fst}_C] = \times_C$ and $t_C[\text{fst}_C] = \text{fst}[C_0, C_0]$
- $s_C[\text{snd}_C] = \times_C$ and $t_C[\text{snd}_C] = \text{snd}[C_0, C_0]$

with the following universal property:

- for any objects $a, b, c : J \to C_0$ and morphisms $f, g : J \to C_1$ (with $J$ arbitrary) such that
  
  $s_C[f] = c$ and $t_C[f] = a$
  
  $s_C[g] = c$ and $t_C[g] = b$

  there is a unique morphism $\langle f, g \rangle_C : J \to C_1$ such that

  $s_C[\langle f, g \rangle_C] = c$
  
  $t_C[\langle f, g \rangle_C] = a \times_C b$
  
  $\text{fst}_C[a, b] \circ_C \langle f, g \rangle_C = f$
  
  $\text{snd}_C[a, b] \circ_C \langle f, g \rangle_C = g$

where we write $a \times_C b$, $\text{fst}_C[a, b]$, $\text{snd}_C[a, b]$ for the arrows $\times_C \circ (a, b)$, $\text{fst}_C \circ (a, b)$, $\text{snd}_C \circ (a, b)$. 20
If $C$ has products, then we have the following:

- for any objects $a, b, c, d : J \to C_0$ and morphisms $f, g : J \to C_1$ such that
  
  \[
  s_C[f] = a \quad \text{and} \quad t_C[f] = c \\
  s_C[g] = b \quad \text{and} \quad t_C[g] = d
  \]

  there exists a unique morphism $f \times_C g : J \to C_1$ such that
  
  \[
  s_C[f \times_C g] = a \times_C b \\
  t_C[f \times_C g] = c \times_C d \\
  \text{fst}_C[c, d] \circ_C (f \times_C g) = f \circ_C \text{fst}_C[a, b] \\
  \text{snd}_C[c, d] \circ_C (f \times_C g) = g \circ_C \text{snd}_C[a, b]
  \]

Using this observation, it is possible to show that above definition is equivalent to the standard one given e.g., in Section 7.2 of [7].

**Definition 43.** A reflexive graph category $\mathcal{X}$ has products if for each $l \in \{0, 1\}$ the category $\mathcal{X}(l)$ has products. The products are stable under face maps if for any $\star \in \text{Bool}$, the canonical morphism witnessing the commutativity of the diagram below is the identity:

\[
\begin{array}{ccc}
\mathcal{X}(1)_0 \times \mathcal{X}(1)_0 & \rightarrow & \mathcal{X}(1)_0 \\
\mathcal{X}(f_\star)_0 \times \mathcal{X}(f_\star)_0 & \downarrow & \mathcal{X}(f_\star)_0 \\
\mathcal{X}(0)_0 \times \mathcal{X}(0)_0 & \rightarrow & \mathcal{X}(0)_0
\end{array}
\]

The products are stable under degeneracies if the canonical morphism $\eta_\mathcal{X}$ witnessing the commutativity of the diagram below is an isomorphism:

\[
\begin{array}{ccc}
\mathcal{X}(0)_0 \times \mathcal{X}(0)_0 & \rightarrow & \mathcal{X}(0)_0 \\
\mathcal{X}(d)_0 \times \mathcal{X}(d)_0 & \downarrow & \mathcal{X}(d)_0 \\
\mathcal{X}(1)_0 \times \mathcal{X}(1)_0 & \rightarrow & \mathcal{X}(1)_0
\end{array}
\]

**Notation 44.** If $\mathcal{X}$ has products stable under degeneracies, we write $\eta_\mathcal{X}[a, b]$ for the composition $\eta_\mathcal{X} \circ (a, b)$ whenever $a, b : J \to \mathcal{X}(0)_0$ are two objects.

If $\mathcal{X}$ has products stable under degeneracies, we have:

- for any objects $a, b : J \to \mathcal{X}(0)_0$, the following diagrams commute:
The isomorphism $\eta_X^* \times X$ is coherent, i.e., for any objects $a, b : J \to X(0)_0$:

$$X(f)_1 \circ \eta_X^*[a, b] = id$$

The isomorphism $\eta_X^* \times X$ is natural, i.e., for any objects $a, b, c, d : J \to X(0)_0$ and morphisms $f, g : J \to X(0)_1$ such that

$$s_{X(0)}[f] = a \quad \text{and} \quad t_{X(0)}[f] = c$$
$$s_{X(0)}[g] = b \quad \text{and} \quad t_{X(0)}[g] = d$$

the following diagram commutes:

$$\begin{array}{ccc}
X(d)_0 \circ (a \times_{X(0)} b) & \xrightarrow{\eta_X^*[a, b]} & (X(d)_0 \circ a) \times_{X(1)} (X(d)_0 \circ a) \\
\downarrow & & \downarrow \\
X(d)_1 \circ \text{fst}_{X(0)}[a, b] & \xrightarrow{\text{fst}_{X(1)}[X(d)_0 \circ a, X(d)_0 \circ b]} & X(d)_0 \circ a
\end{array}$$

$$\begin{array}{ccc}
X(d)_0 \circ (a \times_{X(0)} b) & \xrightarrow{\eta_X^*[a, b]} & (X(d)_0 \circ a) \times_{X(1)} (X(d)_0 \circ a) \\
\downarrow & & \downarrow \\
X(d)_1 \circ \text{snd}_{X(0)}[a, b] & \xrightarrow{\text{snd}_{X(1)}[X(d)_0 \circ a, X(d)_0 \circ b]} & X(d)_0 \circ a
\end{array}$$

Definition 45. A reflexive graph category $(\mathcal{X}, (\mathcal{M}, I))$ with isomorphisms has products if $\mathcal{X}$ has products and for any $f, g : J \to X(0)_1$, $f \times_{X(1)} g$ is in the image of $I(l)$ whenever $f$ and $g$ are. The products are stable under face maps if the products in $\mathcal{X}$ are stable under face maps. The products are stable under degeneracies if the products in $\mathcal{X}$ are stable under degeneracies and the (iso)morphism $\eta_X^*$ is in the image of $I(1)$.

Lemma 46. If a reflexive graph category $(\mathcal{X}, (\mathcal{M}, I))$ with isomorphisms has products stable under face maps and degeneracies, then for each $n$, the category $\mathcal{M}^n \to \mathcal{M}$ has products.

Proof. Fix $\mathcal{F}$ and $\mathcal{G}$ in $\mathcal{M}^n \to \mathcal{M}$. We define $\mathcal{F} \times \mathcal{G}$ by:
The first projection out of \( \mathcal{F} \times \mathcal{G} \) is defined as the reflexive graph natural transformation whose component at level \( l \) is \( \text{fst}_{\mathcal{X}(l)}(\mathcal{F}(l)_0, \mathcal{G}(l)_0) \). To prove naturality – with respect to \( \mathcal{F} \times \mathcal{G} \) and \( \mathcal{G} \) – we observe the following chain of equalities:

\[
\begin{align*}
\text{fst}_{\mathcal{X}(l)}(\mathcal{F}(l)_0, \mathcal{G}(l)_0) \circ t_{\mathcal{M}(l^n)} \circ \eta_{\mathcal{X}(l)} \circ \mathcal{F}(l)_1 \times \mathcal{G}(l)_1 &= \text{fst}_{\mathcal{X}(l)}(\mathcal{F}(l)_0 \circ t_{\mathcal{M}(l^n)}, \mathcal{G}(l)_0 \circ t_{\mathcal{M}(l^n)}) \circ \mathcal{F}(l)_1 \times \mathcal{G}(l)_1 \\
&= \mathcal{F}(l)_1 \circ \text{fst}_{\mathcal{X}(l)}(\mathcal{F}(l)_0 \circ s_{\mathcal{M}(l^n)}, \mathcal{G}(l)_0 \circ s_{\mathcal{M}(l^n)}) \\
&= \mathcal{F}(l)_1 \circ \mathcal{X}(l) \circ (\text{fst}_{\mathcal{X}(l)}(\mathcal{F}(l)_0, \mathcal{G}(l)_0) \circ s_{\mathcal{M}(l^n)})
\end{align*}
\]

The first and third equalities are clear and the second follows by the definition of \( \times_{\mathcal{X}(l)} \) on morphisms. To prove degeneracy-preservation – with respect to \( \varepsilon_{\mathcal{F} \times \mathcal{G}} \) and \( \varepsilon_{\mathcal{G}} \) – we observe the following chain of equalities:

\[
\begin{align*}
\text{fst}_{\mathcal{X}(l)}(\mathcal{F}(l)_0, \mathcal{G}(l)_0) \circ \mathcal{M}(d^n_0) \circ \varepsilon_{\mathcal{X}(l)} \circ \mathcal{X}(l) \circ \mathcal{F}(l)_1 \times \mathcal{G}(l)_1 &= \text{fst}_{\mathcal{X}(l)}(\mathcal{F}(l)_0 \circ \mathcal{M}(d^n_0), \mathcal{G}(l)_0 \circ \mathcal{M}(d^n_0)) \circ \mathcal{X}(l) \circ \mathcal{F}(l)_1 \times \mathcal{G}(l)_1 \\
&= \varepsilon_{\mathcal{F}} \circ \mathcal{X}(l) \circ \text{fst}_{\mathcal{X}(l)}(\mathcal{X}(d)_0 \circ \mathcal{F}(l)_0, \mathcal{X}(d)_0 \circ \mathcal{G}(l)_0) \circ \mathcal{X}(l) \circ \mathcal{F}(l)_1 \times \mathcal{G}(l)_1 \\
&= \varepsilon_{\mathcal{F} \times \mathcal{G}} \circ \mathcal{X}(l) \circ (\text{fst}_{\mathcal{X}(l)}(\mathcal{F}(l)_0, \mathcal{G}(l)_0) \circ s_{\mathcal{M}(l^n)})
\end{align*}
\]

The first equality is clear, the second follows by definition of \( \times_{\mathcal{X}(l)} \) on morphisms, and the third follows by definition of \( \eta^\varnothing_{\mathcal{X}(l)} \). The preservation of face maps follows by the exact same argument. This shows that the first projection is indeed a proper morphism. The second projection is defined analogously.

To show that \( \mathcal{F} \times \mathcal{G} \) with the aforementioned projections is indeed a product, fix \( \mathcal{H} \) and \( \eta_{\mathcal{F}} : \mathcal{H} \to \mathcal{F} \), \( \eta_{\mathcal{G}} : \mathcal{H} \to \mathcal{G} \). The universal morphism from \( \mathcal{H} \) into \( \mathcal{F} \times \mathcal{G} \) is the reflexive graph natural transformation whose component at level \( l \) is \( \langle \eta_{\mathcal{F}}(l), \eta_{\mathcal{G}}(l) \rangle_{\mathcal{X}(l)} \). To show naturality – with respect to \( \mathcal{H} \) and \( \mathcal{F} \times \mathcal{G} \) – we need to establish the equality

\[
\begin{align*}
\langle \eta_{\mathcal{F}}(l), \eta_{\mathcal{G}}(l) \rangle_{\mathcal{X}(l)} \circ \text{fst}_{\mathcal{M}(l^n)} \circ \mathcal{H}(l)_1 &= \langle \mathcal{F}(l)_1 \times_{\mathcal{X}(l)} \mathcal{G}(l)_1 \rangle \circ \mathcal{X}(l) \circ \langle \eta_{\mathcal{F}}(l), \eta_{\mathcal{G}}(l) \rangle_{\mathcal{X}(l)} \circ s_{\mathcal{M}(l^n)}
\end{align*}
\]

The target of the two morphisms is a product, so it suffices to check that their compositions with the first and second projections coincide. The chain of equalities below establishes this for the first projection. Equalities (1) and (5) are clear; equalities (2) and (4) follow by the definition of \( \langle \cdot, \cdot \rangle_{\mathcal{X}(l)} \); equality (3) follows from the naturality of \( \eta_{\mathcal{F}} \); and equality (6) follows by the definition of \( \times_{\mathcal{X}(l)} \) on morphisms. The case of the second projection is entirely analogous.
To prove that our candidate universal morphism is degeneracy-preserving – with respect to \( \varepsilon_H \) and \( \varepsilon_{X \times X} \) – we need to establish the equality

\[
(\langle \eta_F, \eta_G \rangle, \langle \eta_F, \eta_G \rangle) \circ \langle F, G \rangle \varepsilon_{H} =
(\varepsilon_{F \times X}, \varepsilon_{G \times X}) \circ \langle X \times X \rangle \circ \langle X \times X \rangle \circ \langle X \times X \rangle \circ \langle \eta_F, \eta_G \rangle \circ \langle \eta_F, \eta_G \rangle)
\]

Again the target of the two morphisms is a product so it suffices to check that their compositions with the first and second projections coincide. The chain of equalities below establishes this for the first projection. Equality (1) is clear; equalities (2) and (4) follow by the definition of \( \langle \cdot, \cdot \rangle_{X(1)} \); equality (3) follows by the degeneracy-preservation of \( \eta_F \); equality (5) follows by the functoriality of \( \chi(d) \); equality (6) follows by the definition of \( \eta_X \); and equality (7) follows by the definition of \( \times_{X(1)} \) on morphisms. The case of the second projection is entirely analogous, which shows that our candidate universal morphism is degeneracy-preserving. The preservation of face maps is shown by the exact same argument. Thus our candidate universal morphism is indeed a proper morphism. Its universality and uniqueness are obvious, again by the universal property of \( \times_{X(1)} \).

\[
\text{fst}_{X(1)} [F(l)_0 \circ t_{M(0)}^\cdot, G(l)_0 \circ t_{M(0)}^\cdot] \circ \chi(1) \circ \langle \eta_F(l), \eta_G(l) \rangle \circ t_{M(0)}^\cdot \circ \chi(1) \circ \varepsilon_{X(1) \times X(1)}
\]

\[
\begin{align*}
(1) & \quad \left( \text{fst}_{X(1)} [F(l)_0 \circ t_{M(0)}^\cdot, G(l)_0 \circ t_{M(0)}^\cdot] \circ \chi(1) \circ \langle \eta_F(l), \eta_G(l) \rangle \circ t_{M(0)}^\cdot \circ \chi(1) \circ \varepsilon_{X(1) \times X(1)} \right) \\
(2) & \quad \langle \eta_F(l) \circ t_{M(0)}^\cdot \rangle \circ \chi(1) \circ \varepsilon_{X(1) \times X(1)} \\
(3) & \quad F(l)_1 \circ \chi(1) \circ \varepsilon_{X(1) \times X(1)} \\
(4) & \quad \text{fst}_{X(1)} [F(l)_0 \circ t_{M(0)}^\cdot, G(l)_0 \circ t_{M(0)}^\cdot] \circ \chi(1) \circ \langle \eta_F(l), \eta_G(l) \rangle \circ t_{M(0)}^\cdot \\
(5) & \quad F(l)_1 \circ \chi(1) \circ \langle \eta_F(l), \eta_G(l) \rangle \circ t_{M(0)}^\cdot \\
(6) & \quad \text{fst}_{X(1)} [F(l)_0 \circ t_{M(0)}^\cdot, G(l)_0 \circ t_{M(0)}^\cdot] \circ \chi(1) \circ \langle \eta_F(l), \eta_G(l) \rangle \circ t_{M(0)}^\cdot \\
(7) & \quad \chi(1) \circ \langle \eta_F(l), \eta_G(l) \rangle \circ \varepsilon_{X(1) \times X(1)} \\
\end{align*}
\]
Definition 47. An internal category \( C \) in \( C \) with products has exponentials if it comes equipped with arrows \( \Rightarrow_C : C_0 \times C_0 \to C_0 \) and \( \text{eval}_C : C_0 \times C_0 \to C_1 \) such that

\[
s_C[\text{eval}_C] = ((\Rightarrow_C) \times_C \text{fst}(C_0, C_0)) \quad \text{and} \quad t_C[\text{eval}_C] = \text{snd}(C_0, C_0)
\]

with the following universal property:

- for any objects \( a, b, c : J \to C_0 \) and morphism \( f : J \to C_1 \) (with \( J \) arbitrary) such that

\[
s_C[f] = c \times_C a \quad \text{and} \quad t_C[f] = b
\]

there is a unique morphism \( \lambda_C[a, b, c, f] : J \to C_1 \) such that

\[
s_C[\lambda_C[a, b, c, f]] = c \quad \text{and} \quad t_C[\lambda_C[a, b, c, f]] = (a \Rightarrow_C b)
\]

\[\text{eval}_C[a, b] \circ_C (\lambda_C[a, b, c, f] \times_C \text{id}_C[a]) = f\]

where we write \( a \Rightarrow_C b \), \( \text{eval}_C[a, b] \) for the arrows \((\Rightarrow_C \circ (a, b)), \text{eval}_C \circ (a, b)\).

If \( C \) has exponentials, then we have the following:

- for any objects \( a, b, c, d : J \to C_0 \) and morphisms \( f, g : J \to C_1 \) such that

\[
s_C[f] = c \times_C a \quad \text{and} \quad t_C[f] = a
\]

\[
s_C[g] = b \quad \text{and} \quad t_C[g] = d
\]

there exists a unique morphism \( f \Rightarrow_C g : J \to C_1 \) such that

\[
s_C[f \Rightarrow_C g] = (a \Rightarrow_C b) \quad \text{and} \quad t_C[f \Rightarrow_C g] = (c \Rightarrow_C d)
\]

\[\text{eval}_C[a, d] \circ_C ((f \Rightarrow_C g) \times_C \text{id}_C[c]) =
\]

\[g \circ_C \text{eval}_C[a, b] \circ_C (\text{id}_C[a \Rightarrow_C b] \times_C f)\]

Using this observation, it is possible to show that above definition is equivalent to the standard one given e.g., in Section 7.2 of [7].

Definition 48. A reflexive graph category \( \mathcal{X} \) with products has exponentials if for each \( l \in \{0, 1\} \), the category \( \mathcal{X}(l) \) has exponentials. Assuming the products are stable under face maps, we say the exponentials are stable under face maps if for any \( \ast \in \text{Bool} \), the canonical morphism witnessing the commutativity of the diagram below is the identity:
Assuming the products are stable under degeneracies, we say the exponentials are stable under degeneracies if the canonical morphism $\eta \Rightarrow X$ witnessing the commutativity of the diagram below is an isomorphism:

$\begin{array}{ccc}
\mathcal{X}(0) \times \mathcal{X}(0) & \Rightarrow \mathcal{X}(0) \\
\mathcal{X}(1) \times \mathcal{X}(1) & \Rightarrow \mathcal{X}(1) \\
\mathcal{X}(d) \times \mathcal{X}(d) & \Rightarrow \mathcal{X}(d)
\end{array}$

**Notation 49.** If $\mathcal{X}$ has exponentials stable under degeneracies, we write $\eta \Rightarrow X[a,b]$ for the composition $\eta \circ \langle a, b \rangle$ whenever $a, b : J \rightarrow \mathcal{X}(0)$ are two objects.

If $\mathcal{X}$ has products and exponentials stable under degeneracies, we have:

- for any objects $a, b : J \rightarrow \mathcal{X}(0)$, the following diagram commutes:

$\begin{array}{ccc}
\mathcal{X}(d) \circ \text{eval}_{\mathcal{X}(0)}[a,b] & \Rightarrow \mathcal{X}(d) \circ \text{id}_{\mathcal{X}(0)}[a,b] \\
\mathcal{X}(d) \circ \text{id}_{\mathcal{X}(0)}[a,b] & \Rightarrow \mathcal{X}(d) \circ \text{id}_{\mathcal{X}(0)}[a,b]
\end{array}$

- The isomorphism $\eta \Rightarrow X$ is coherent, i.e., for any objects $a, b : J \rightarrow \mathcal{X}(0)$, the following holds:

$\mathcal{X}(f) \circ \eta[a,b] = \text{id}$

- The isomorphism $\eta \Rightarrow X$ is natural, i.e., for any objects $a, b, c, d : J \rightarrow \mathcal{X}(0)$ and morphisms $f, g : J \rightarrow \mathcal{X}(0)$ such that

$\begin{align*}
\text{s}_{\mathcal{X}(0)}[f] &= a \\
\text{t}_{\mathcal{X}(0)}[f] &= c \\
\text{s}_{\mathcal{X}(0)}[g] &= b \\
\text{t}_{\mathcal{X}(0)}[g] &= d
\end{align*}$

the following diagram commutes:
Definition 50. A reflexive graph category \((\mathcal{X}, (\mathcal{M}, I))\) with isomorphisms and products has exponentials if \(\mathcal{X}\) has exponentials and for any \(f, g : J \to \mathcal{X}(l)\), \(f \Rightarrow \mathcal{X}(l) g\) is in the image of \(I(l)\) whenever \(f\) and \(g\) are. Assuming the products are stable under face maps, we say the exponentials are stable under face maps if the exponentials in \(\mathcal{X}\) are stable under face maps. Assuming the products are stable under degeneracies, we say the exponentials are stable under degeneracies if the exponentials in \(\mathcal{X}\) are stable under degeneracies and the (iso)morphism \(\eta_{\mathcal{X}}\) is in the image of \(I(1)\).

Lemma 51. If a reflexive graph category \((\mathcal{X}, (\mathcal{M}, I))\) with isomorphisms has products and exponentials stable under face maps and degeneracies, then for each \(n\), the category \(\mathcal{M}^n \to \mathcal{M}\) has exponentials.

Proof. Fix \(\mathcal{F}\) and \(\mathcal{G}\) in \(\mathcal{M}^n \to \mathcal{M}\). We define \(\mathcal{F} \Rightarrow \mathcal{G}\) by:

- \((\mathcal{F} \Rightarrow \mathcal{G})(l)_0 := \mathcal{F}(l)_0 \Rightarrow \mathcal{X}(l) \mathcal{G}(l)_0\)
- \((\mathcal{F} \Rightarrow \mathcal{G})(l)_1 := \mathcal{F}(l)_1^{-1} \Rightarrow \mathcal{X}(l) \mathcal{G}(l)_1\)
- \(\varepsilon_{\mathcal{F} \Rightarrow \mathcal{G}} := (\varepsilon_{\mathcal{F})^{-1} \Rightarrow \mathcal{X}(l) \varepsilon_{\mathcal{G}}) \circ_{\mathcal{M}(1)} \eta_{\mathcal{X}}(0, \mathcal{G}(0))\)

The evaluation morphism for \(\mathcal{F} \Rightarrow \mathcal{G}\) is defined as the reflexive graph natural transformation whose component at level \(l\) is \(\text{eval}_{\mathcal{X}(l)}[\mathcal{F}(l)_0, \mathcal{G}(l)_0]\). To prove naturality – with respect to \((\mathcal{F} \Rightarrow \mathcal{G}) \times \mathcal{F}\) and \(\mathcal{G}\) – we observe the chain of equalities below. The first and third equalities follow since \(\times_{\mathcal{X}(1)}\) suitably commutes with \(\circ_{\mathcal{X}(1)}\); the second equality follows by definition of \(\Rightarrow_{\mathcal{X}(1)}\) on morphisms; and the fourth equality follows since the product of identities is again an identity.
(eval_{X(1)}[F(l)_0, G(l)_0] \circ t_{M(l)_0^n}) \circ_{X(1)} \left( (F(l)_1^{-1} \Rightarrow X(l) \ G(l)_1) \times_{X(l)} F(l)_1 \right)

(1) eval_{X(1)}[F(l)_0 \circ t_{M(l)_0^n}, G(l)_0 \circ t_{M(l)_0^n}] \circ_{X(1)} \left( (F(l)_1^{-1} \Rightarrow X(l) \ G(l)_1) \times_{X(l)} id_{X(1)}[F(l)_0 \circ t_{M(l)_0^n}] \right) \circ_{X(1)} \left( id_{X(1)}[F(l)_0 \circ s_{M(l)_0^n}] \Rightarrow X(l) \ (G(l)_0 \circ s_{M(l)_0^n}) \right) \times_{X(l)} F(l)_1

(2) G(l)_1 \circ_{X(l)} eval_{X(1)}[F(l)_0 \circ s_{M(l)_0^n}, G(l)_0 \circ s_{M(l)_0^n}] \circ_{X(l)} \left( id_{X(1)}[F(l)_0 \circ s_{M(l)_0^n}] \Rightarrow X(l) \ (G(l)_0 \circ s_{M(l)_0^n}) \right) \times_{X(l)} id_{X(1)}[F(l)_0 \circ s_{M(l)_0^n}]

(3) G(l)_1 \circ_{X(l)} (eval_{X(1)}[F(l)_0, G(l)_0] \circ s_{M(l)_0^n}) \circ_{X(l)} \left( id_{X(1)}[F(l)_0 \circ s_{M(l)_0^n}] \Rightarrow X(l) \ (G(l)_0 \circ s_{M(l)_0^n}) \right)

(4) G(l)_1 \circ_{X(l)} (eval_{X(1)}[F(l)_0, G(l)_0] \circ s_{M(l)_0^n})

To prove the degeneracy-preservation of the evaluation morphism – with respect to $\varepsilon_{(F \Rightarrow G) \times F}$ and $\varepsilon_{G}$ – we observe the chain of equalities below.

$$(eval_{X(1)}[F(1)_0, G(1)_0] \circ M(d)_0^n) \circ_{X(1)}$$

$$(\varepsilon_{F}^{-1} \Rightarrow X(1) \ \varepsilon_G) \circ_{X(1)} \eta_{X}^{\varepsilon} [F(0)_0, G(0)_0] \times_{X(1)} \varepsilon_{F} \circ_{X(1)}$$

$$(1) \ \eta_{X}^{\varepsilon} [F(0)_0 \Rightarrow X(0) \ G(0)_0, F(0)_0]$$

$$(2) \ \varepsilon_{F} \circ_{X(1)} \ \varepsilon_{F} \circ_{X(1)} [X(d)_0 \circ F(0)_0, X(d)_0 \circ G(0)_0] \circ_{X(1)}$$

$$(id_{X(1)}[X(d)_0 \circ F(0)_0] \Rightarrow X(1) \ X(d)_0 \circ G(0)_0] \times_{X(1)} \varepsilon_{F}^{-1}) \circ_{l_2}$$

$$(\eta_{X}^{\varepsilon} [F(0)_0, G(0)_0] \times_{X(1)} \varepsilon_{F} \circ_{X(1)}$$

$$(3) \ \eta_{X}^{\varepsilon} [F(0)_0 \Rightarrow X(0) \ G(0)_0]$$

$$(4) \ \varepsilon_{F} \circ_{X(1)} [X(d)_1 \circ eval_{X(0)}[F(0)_0, G(0)_0])$$

The first and third equalities follow since $\times_{X(1)}$ suitably commutes with $\circ_{X(1)}$; the second equality follows by definition of $\Rightarrow_{X(1)}$ on morphisms; and the fourth equality
follows by definition of $\eta_X$. The preservation of face maps follows by the exact same argument. This shows that the evaluation morphism is indeed a proper morphism.

To show that $F \Rightarrow G$ with the aforementioned evaluation is an exponential, fix $H$ and $\eta : F \times H \to G$. The universal morphism from $H$ into $F \Rightarrow G$ is the reflexive graph

natural transformation whose component at level $l$ is $\lambda_X(l)[F(l)_0, G(l)_0, \eta(l)]$.

To show naturality – with respect to $H$ and $F \Rightarrow G$ – we need to establish the equality

$$(\lambda_X(l)[F(l)_0, G(l)_0, H(l)_0, \eta(l)] \circ t_{M(l)_n}) \circ_X(l) H(l)_1 =$$

$$(F(l)_1^{-1} \Rightarrow_X(l) G(l)_1) \circ_X(l) (\lambda_X(l)[F(l)_0, G(l)_0, H(l)_0, \eta(l)] \circ s_{M(l)_n})$$

The target of the two morphisms is an exponential, so it suffices to check that taking a product of each morphism with the identity and postcomposing with the evaluation morphism yields the same result. Moreover, since $\text{id}_{X(l)[H(l)_0 \circ s_{M(l)_n}] \times X(l)} F(l)_1$ is an isomorphism, it suffices to show that a further precomposition with this isomorphism yields the same result. To this end we observe the chain of equalities below. Equalities (1), (5), (7), (8), and the green part of (2) follow since $\times_X(l)$ suitably commutes with $\circ_X(l)$; equality (4) and the red part of (2) follow by the definition of $\lambda_X(l)$; equality (3) follows by the degeneracy-preservation of $\eta$; and equality (6) follows by the definition of $\Rightarrow_X(l)$.
\[\text{eval}_X[l](\mathcal{F}(l)_0 \circ t_{M(l)^n}, \mathcal{G}(l)_0 \circ t_{M(l)^n}) \circ X(l)\]

\[\left(\left((\lambda X(l)[\mathcal{F}(l)_0, \mathcal{G}(l)_0, \mathcal{H}(l)_0, \eta(l)] \circ t_{M(l)^n}) \circ X(l) \mathcal{H}(l)_1 \right) \times X(l)\right)\]

\[\text{id}_{X(l)}[\mathcal{F}(l)_0 \circ t_{M(l)^n}] \circ X(l)(\text{id}_{X(l)}[\mathcal{H}(l)_0 \circ s_{M(l)^n}] \times X(l) \mathcal{F}(l)_1)\]

\[\text{eval}_{X(l)}[\mathcal{F}(l)_0 \circ t_{M(l)^n}, \mathcal{G}(l)_0 \circ t_{M(l)^n}] \circ X(l)\]

\[\left((\lambda X(l)[\mathcal{F}(l)_0, \mathcal{G}(l)_0, \mathcal{H}(l)_0, \eta(l)] \circ t_{M(l)^n}) \times X(l) \text{id}_{X(l)}[\mathcal{F}(l)_0 \circ t_{M(l)^n}]\right) \circ X(l)\]

\[\left([\mathcal{H}(l)_1 \times X(l) \text{id}_{X(l)}[\mathcal{F}(l)_0 \circ t_{M(l)^n}]] \circ X(l) \left([\text{id}_{X(l)}[\mathcal{H}(l)_0 \circ s_{M(l)^n}] \times X(l) \mathcal{F}(l)_1\right)\right)\]

\[\left((\lambda X(l)[\mathcal{F}(l)_0, \mathcal{G}(l)_0, \mathcal{H}(l)_0, \eta(l)] \circ s_{M(l)^n}\right) \times X(l) \mathcal{F}(l)_1\]

\[\text{id}_{X(l)}[\mathcal{F}(l)_0 \circ t_{M(l)^n}] \circ X(l)(\text{id}_{X(l)}[\mathcal{H}(l)_0 \circ s_{M(l)^n}] \times X(l) \mathcal{F}(l)_1)\]

\[\text{eval}_{X(l)}[\mathcal{F}(l)_0 \circ t_{M(l)^n}, \mathcal{G}(l)_0 \circ t_{M(l)^n}] \circ X(l)\]

\[\left((\mathcal{F}(l)_1^{-1} \Rightarrow X(l) \mathcal{G}(l)_1) \times X(l) \text{id}_{X(l)}[\mathcal{F}(l)_0 \circ t_{M(l)^n}]\right) \circ X(l)\]

\[\left(([\mathcal{F}(l)_1^{-1} \Rightarrow X(l) \mathcal{G}(l)_1] \times X(l) \text{id}_{X(l)}[\mathcal{F}(l)_0 \circ t_{M(l)^n}]\right) \circ X(l)\]

\[\left([\text{id}_{X(l)}[\mathcal{H}(l)_0 \circ s_{M(l)^n}] \times X(l) \mathcal{F}(l)_1\right)\]

\[\left((\lambda X(l)[\mathcal{F}(l)_0, \mathcal{G}(l)_0, \mathcal{H}(l)_0, \eta(l)] \circ s_{M(l)^n}\right) \times X(l) \mathcal{F}(l)_1\]

\[\left([\text{id}_{X(l)}[\mathcal{H}(l)_0 \circ s_{M(l)^n}] \times X(l) \mathcal{F}(l)_1\right)\]
To prove that our candidate universal morphism is degeneracy-preserving – with respect to $\varepsilon_H$ and $\varepsilon_F \Rightarrow \mathcal{G}$ – we need to establish the following equality:

$$\left( \lambda_{X(1)}[\mathcal{F}(1)_0, \mathcal{G}(1)_0, \mathcal{H}(1)_0, \eta(1)] \circ \mathcal{M}(d)_0^0 \right) \circ_{X(1)} \varepsilon_H =
$$

$$\left( \varepsilon_F^{-1} \Rightarrow_{X(1)} \varepsilon_F \right) \circ_{X(1)} \eta_X^X \left[ \mathcal{F}(0)_0, \mathcal{G}(0)_0 \right] \circ_{X(1)}
$$

$$(\mathcal{X}(d)_{1} \circ \lambda_{X(0)}[\mathcal{F}(0)_0, \mathcal{G}(0)_0, \mathcal{H}(0)_0, \eta(0)])$$

The target of the two morphisms is an exponential, so it suffices to check that taking a product of each morphism with the identity and postcomposing with the evaluation morphism yields the same result. Moreover, since

$$(\text{id}_{X(1)}[\mathcal{X}(d)]_0 \circ \mathcal{H}(0)_0 \times_{X(1)} \varepsilon_F) \circ_{X(1)} \eta_X^X \left[ \mathcal{F}(0)_0, \mathcal{H}(0)_0 \right]$$

is an isomorphism, it suffices to show that a further precomposition with this isomorphism yields the same result. To this end we observe the two chains of equalities below. Equalities (1), (7), (9), (10), and the green part of (2) (the red part of equality (4) and the red part of (2) follow by the definition of $\lambda_{X(1)}$; equality (3) follows by the degeneracy-preservation of $\eta$; equality (5) follows by the functoriality of $\mathcal{X}(d)$; equality (6) follows by the definition of $\eta_X^X$; the orange part of equality (8) follows by the definition of $\Rightarrow_{X(1)}$ on morphisms; and the purple part of equality (8) follows by the naturality of $\eta_X^X$. The preservation of face maps is shown by the exact same argument.

This shows that our candidate universal morphism is a proper morphism. Its universality and uniqueness are obvious by the universal property of $\Rightarrow_{X(1)}$.

\begin{align*}
\text{eval}_{X(1)}[\mathcal{F}(1)_0 \circ \mathcal{M}(d)_0^0, \mathcal{G}(1)_0 \circ \mathcal{M}(d)_0^0] \circ_{X(1)} \\
\left( \left( \lambda_{X(1)}[\mathcal{F}(1)_0, \mathcal{G}(1)_0, \mathcal{H}(1)_0, \eta(1)] \circ \mathcal{M}(d)_0^0 \right) \circ_{X(1)} \varepsilon_H \right) \\
\times_{X(1)} \text{id}_{X(1)}[\mathcal{F}(1)_0 \circ \mathcal{M}(d)_0^0] \circ_{X(1)} \eta_X^X \left[ \mathcal{F}(0)_0, \mathcal{H}(0)_0 \right]
\end{align*}

$$(1)$$

\begin{align*}
\text{eval}_{X(1)}[\mathcal{F}(1)_0 \circ \mathcal{M}(d)_0^0, \mathcal{G}(1)_0 \circ \mathcal{M}(d)_0^0] \circ_{X(1)} \\
\left( \left( \lambda_{X(1)}[\mathcal{F}(1)_0, \mathcal{G}(1)_0, \mathcal{H}(1)_0, \eta(1)] \circ \mathcal{M}(d)_0^0 \right) \times_{X(1)} \text{id}_{X(1)}[\mathcal{F}(1)_0 \circ \mathcal{M}(d)_0^0] \right) \\
\times_{X(1)} \text{eval}_{X(1)}[\mathcal{F}(0)_0 \circ \mathcal{M}(d)_0^0] \circ_{X(1)} \eta_X^X \left[ \mathcal{F}(0)_0, \mathcal{H}(0)_0 \right]
\end{align*}

$$(2)$$

\begin{align*}
(\eta(1) \circ \mathcal{M}(d)_0^0) \circ_{X(1)} \left( \varepsilon_H \times_{X(1)} \varepsilon_F \right) \circ_{X(1)} \eta_X^X \left[ \mathcal{F}(0)_0, \mathcal{H}(0)_0 \right]
\end{align*}

$$(3)$$

\begin{align*}
\varepsilon_F \circ_{X(1)} (\mathcal{X}(d)_{1} \circ \eta(0))
\end{align*}

$$(4)$$

\begin{align*}
\varepsilon_F \circ_{X(1)} \left( \mathcal{X}(d)_{1} \circ \left( \text{eval}_{X(0)}[\mathcal{F}(0)_0, \mathcal{G}(0)_0] \circ_{X(0)} \right) \\
\left( \lambda_{X(0)}[\mathcal{F}(0)_0, \mathcal{G}(0)_0, \mathcal{H}(0)_0, \eta(0)] \times_{X(0)} \text{id}_{X(0)}[\mathcal{F}(0)_0] \right) \right)
\end{align*}

$$(5)$$
\[
\varepsilon_G \circ \chi(1) \left( \chi(d)_1 \circ \text{eval}_{X(0)}[\mathcal{F}(0), \mathcal{G}(0)] \right) \circ \chi(1)
\]
\[
\left( \chi(d)_1 \circ (\lambda_{\mathcal{X}(0)}[\mathcal{F}(0), \mathcal{G}(0), \mathcal{H}(0), \eta(0)] \times \chi(0) \circ \text{id}_{\mathcal{X}(0)}[\mathcal{F}(0)]_0) \right) \circ \chi(1)
\]
\[
\text{eval}_{\mathcal{Y}(0)} \left[ \mathcal{X}(d)_0 \circ \mathcal{F}(0)_0, \chi(d)_0 \circ \mathcal{G}(0)_0 \right] \circ \chi(1)
\]
\[
\left( \eta^\mathcal{X}_n[\mathcal{F}(0), \mathcal{G}(0)] \times \chi(1) \circ \text{id}_{\mathcal{X}(0)}[\mathcal{F}(0)]_0 \circ \mathcal{F}(0)_0 \right) \circ \chi(1)
\]
\[
\left( \chi(d)_1 \circ (\lambda_{\mathcal{X}(0)}[\mathcal{F}(0), \mathcal{G}(0), \mathcal{H}(0), \eta(0)] \times \chi(0) \circ \text{id}_{\mathcal{X}(0)}[\mathcal{F}(0)]_0) \right) \circ \chi(1)
\]
\[
\text{eval}_{\mathcal{Y}(0)} \left[ \mathcal{X}(d)_0 \circ \mathcal{F}(0)_0, \chi(d)_0 \circ \mathcal{G}(0)_0 \right] \circ \chi(1)
\]
\[
\left( \eta^\mathcal{X}_n[\mathcal{F}(0), \mathcal{G}(0)] \times \chi(1) \circ \text{id}_{\mathcal{X}(0)}[\mathcal{F}(0)]_0 \circ \mathcal{F}(0)_0 \right) \circ \chi(1)
\]
\[
\left( \chi(d)_1 \circ (\lambda_{\mathcal{X}(0)}[\mathcal{F}(0), \mathcal{G}(0), \mathcal{H}(0), \eta(0)] \times \chi(0) \circ \text{id}_{\mathcal{X}(0)}[\mathcal{F}(0)]_0) \right) \circ \chi(1)
\]
\[
\text{eval}_{\mathcal{Y}(0)} \left[ \mathcal{X}(d)_0 \circ \mathcal{F}(0)_0, \chi(d)_0 \circ \mathcal{G}(0)_0 \right] \circ \chi(1)
\]
\[
\left( \eta^\mathcal{X}_n[\mathcal{F}(0), \mathcal{G}(0)] \times \chi(1) \circ \text{id}_{\mathcal{X}(0)}[\mathcal{F}(0)]_0 \circ \mathcal{F}(0)_0 \right) \circ \chi(1)
\]
\[
\text{eval}_{\mathcal{Y}(0)} \left[ \mathcal{X}(d)_0 \circ \mathcal{F}(0)_0, \chi(d)_0 \circ \mathcal{G}(0)_0 \right] \circ \chi(1)
\]
\[
\left( \eta^\mathcal{X}_n[\mathcal{F}(0), \mathcal{G}(0)] \times \chi(1) \circ \text{id}_{\mathcal{X}(0)}[\mathcal{F}(0)]_0 \circ \mathcal{F}(0)_0 \right) \circ \chi(1)
\]
\[
\left( \chi(d)_1 \circ (\lambda_{\mathcal{X}(0)}[\mathcal{F}(0), \mathcal{G}(0), \mathcal{H}(0), \eta(0)] \times \chi(0) \circ \text{id}_{\mathcal{X}(0)}[\mathcal{F}(0)]_0) \right) \circ \chi(1)
\]
\[
\text{eval}_{\mathcal{Y}(0)} \left[ \mathcal{X}(d)_0 \circ \mathcal{F}(0)_0, \chi(d)_0 \circ \mathcal{G}(0)_0 \right] \circ \chi(1)
\]
\[
\left( \eta^\mathcal{X}_n[\mathcal{F}(0), \mathcal{G}(0)] \times \chi(1) \circ \text{id}_{\mathcal{X}(0)}[\mathcal{F}(0)]_0 \circ \mathcal{F}(0)_0 \right) \circ \chi(1)
\]
\[
\left( \chi(d)_1 \circ (\lambda_{\mathcal{X}(0)}[\mathcal{F}(0), \mathcal{G}(0), \mathcal{H}(0), \eta(0)] \times \chi(0) \circ \text{id}_{\mathcal{X}(0)}[\mathcal{F}(0)]_0) \right) \circ \chi(1)
\]
\[
\text{eval}_{\mathcal{Y}(0)} \left[ \mathcal{X}(d)_0 \circ \mathcal{F}(0)_0, \chi(d)_0 \circ \mathcal{G}(0)_0 \right] \circ \chi(1)
\]
\[
\left( \eta^\mathcal{X}_n[\mathcal{F}(0), \mathcal{G}(0)] \times \chi(1) \circ \text{id}_{\mathcal{X}(0)}[\mathcal{F}(0)]_0 \circ \mathcal{F}(0)_0 \right) \circ \chi(1)
\]
Definition 52. A reflexive graph category with isomorphisms is cartesian closed if it has terminal objects, products, and exponentials, all stable under face maps and degeneracies.

Example 53. [PER model, continued] Terminal objects, products, and exponentials are defined for \( R_{PER} \) in the obvious ways, inheriting from the corresponding constructs on PERs. It is not hard to check that all of these constructs are preserved on the nose by the two face maps (projections) and the degeneracy (equality functor), and thus, in our terminology, are stable under face maps and degeneracies.

Example 54. [Both versions of Reynolds’ model, continued] Here, too, terminal objects, products, and exponentials are defined for \( R_{REY} \) and \( R_{CREY} \) in the obvious ways, relating two pairs iff their first and second components are related, and two functions iff they map related arguments to related results. It is easy to see that all of these constructs are preserved on the nose (i.e., up to definitional equality) by the projections, and thus are stable under face maps. Unlike in the PER model though, they are only preserved by the equality functor \( \text{Eq} \) up to (the canonical) isomorphism. For example, as discussed just after Definition 11, the two types \( \text{Id}((a, b), (c, d)) \) and \( \text{Id}(a, c) \times \text{Id}(b, d) \) for \( (a, b), (c, d) : A \times B \) are not necessarily identical, although they are isomorphic under the canonical (iso)morphism \( \eta^\times[A, B] : \text{Eq}(A \times B) \to \text{Eq}(A) \times \text{Eq}(B) \). A similar situation arises for function types \( A \to B \): by function extensionality, \( \text{Id}(f, g) \) and \( \Pi_{a, a': A}\text{Id}(f(a), g(a')) \) are isomorphic, but not necessarily identical, via \( \eta^\to[A, B] \). Nevertheless, we still get stability under degeneracies since we explicitly allowed for this possibility in Definition 50.

5 Reflexive Graph Models of Parametricity

As Examples 53 and 54 show, cartesian closed reflexive graph categories with isomorphisms suitably generalize the structure of sets and relations. Moreover, they allow us to interpret unit, product, and function types in a natural way. To show this, we introduce the following terminology, presented in a form more general than we need for interpreting the simply-typed fragment of System F but paralleling the later terminology used for interpreting the impredicative fragment.

Definition 55. A \( \lambda^+ \)-fibration is a split fibration \( U : E \to B \) satisfying the following properties:

1. \( U \) has a split generic object \( \Omega \) in \( B \).

2. \( B \) has a terminal object and products \((-) \times \Omega\), and for every object \( I \) in \( B \), we have \( I \cong \Omega^n \) for some \( n \in \mathbb{N} \).

3. Every fiber \( E_I \) for \( I \) in \( B \) is cartesian closed, with a terminal object \( 1_I \), products \( \times_1 \), and exponentials \( \Rightarrow_1 \).

4. Beck-Chevalley: for any morphism \( f : I \to J \) in \( B \) and objects \( X, Y \) in \( E_J \), the
canonical morphisms below are isomorphisms:

\[ \theta_1(f) : f^*(1_J) \to 1_I \]
\[ \theta_n(f, X, Y) : f^*(X \times_J Y) \to (f^*(X) \times_I f^*(Y)) \]
\[ \theta \Rightarrow (f, X, Y) : f^*(X \Rightarrow_J Y) \to (f^*(X) \Rightarrow_I f^*(Y)) \]

A \( \lambda^\rightarrow \)-fibration is split if these canonical morphisms are identities.

Using a similar idea as in the proof of Lemma 5.2.4 of [7], we can show:

**Lemma 56.** Every \( \lambda^\rightarrow \)-fibration is equivalent to a split \( \lambda^\rightarrow \)-fibration in a canonical way.

We now come to our main technical lemma:

**Theorem 57.** Given a cartesian closed reflexive graph category \( \mathcal{R} \) with isomorphisms, the forgetful functor from the category \( \int_n \mathcal{M}^n \to \mathcal{M} \) to \( \operatorname{Ctx}(\mathcal{R}) \) is a split \( \lambda^\rightarrow \)-fibration.

To interpret \( \forall \)-types we need to know that, in the forgetful fibration from Lemma 57, each weakening functor induced by the first projection from \( n + 1 \) to \( n \) for \( n \in \mathbb{N} \) has a right adjoint \( \forall_n \). Here we differ from [2], where only \( \forall_0 \) is required, with the intention that \( \forall_n \) can be derived from \( \forall_0 \) using partial application. We observe that this approach does not appear to work since a partial application of an indexed functor is not necessarily an indexed functor. Hence we require an entire family of adjoints \( \forall_n \).

**Example 58** (PER model, continued). Define the adjoint \( \forall_n \) by

\[ \forall_n \mathcal{F}(0) \mathcal{A} := \{ (m, k) \mid \text{for all } A, (m, k) \in \mathcal{F}(0) (\mathcal{A}, A), \]
\[ \text{and for all } R, \langle m, k \rangle \in \mathcal{F}(1) (\mathcal{E}q A, R) \} \]
\[ \forall_n \mathcal{F}(1) R := \left( \forall_n \mathcal{F}(0) R^d, \forall_n \mathcal{F}(0) R^c \right), \]
\[ \{ m \mid \text{for all } R, m \in \mathcal{F}(1) (R, R) \} \]

where for any relation \( R := ((A_d, A_c), R_A) \) we write \( R_d \) for \( A_d \) and \( R_c \) for \( A_c \). We will employ a similar convention for Reynolds’ model. To define \( \forall_n \) on a morphism \( \eta : \mathcal{F} \to \mathcal{G} \), we put

\[ \forall_n \eta(0) \mathcal{A} := \left( \forall_n \mathcal{F}(0) \mathcal{A}, \forall_n \mathcal{G}(0) \mathcal{A} \right), \{ m \cdot 0 \} (\forall_n \mathcal{F}(0) \mathcal{A}) \to (\forall_n \mathcal{G}(0) \mathcal{A}) \]

Here \( m \) is any natural number realizing \( \eta(0) \mathcal{A} \). Crucial observations are that all natural transformations are “uniformly realized” in the sense that there is a natural number realizing each such transformation, and since all PERs are defined to be realized by all natural numbers, each is suitably uniform. In particular, if \( \eta \) were not uniformly realized in the above sense then \( \forall_n \eta \) would not be well-defined on morphisms. These observations can be used to show that, in the category-theoretic setting (rather than the setting of \( \omega \)-sets), the family of adjoints \( \forall \) cannot exist precisely because ad hoc natural transformations — i.e., natural transformations that are not uniformly realizable, even though each of their components may indeed be realizable — are not excluded.
Example 59 (Reynolds’ model, continued). On the set level, the adjoint \( \forall_n \) is defined as follows:

\[
\forall_n \mathcal{F}(0) \mathcal{A} := \left\{ f_0 : \Pi_{A \to U} \mathcal{F}(0)(\mathcal{A}, A) \land f_1 : \Pi_{R \to R_0} \mathcal{F}(1)(\mathcal{A}, R)(f_0(0), f_0(1)) \right\}
\]

On the relation level, we define \( \forall_n \mathcal{F}(1) \mathcal{R} \) to be the relation with domain \( \forall_n \mathcal{F}(0) \mathcal{R}_0 \) and codomain \( \forall_n \mathcal{F}(0) \mathcal{R}_1 \) mapping \( ((f_0, f_1), (g_0, g_1)) \) to

\[
\Pi_{R \to R_0} \mathcal{F}(1)(\mathcal{R}, R)(f_0(0), g_0(0))
\]

To see that the above definition indeed gives a degeneracy-preserving reflexive graph functor, fix \( \mathcal{A} \). We want to show that the two relations \( \mathcal{F}(\forall_n \mathcal{F}(0) \mathcal{A}) \) and \( \mathcal{F}(\forall_n \mathcal{F}(1) \mathcal{R}) \) are isomorphic. The domains and codomains of these relations are all the same — \( \forall_n \mathcal{F}(0) \mathcal{A} \) — so we let both of the underlying maps of the isomorphism be identities (as also required by the coherence condition on the isomorphism and, independently, the definition of a relevant isomorphism). Fix \( ((f_0, f_1), (g_0, g_1)) : (\forall_n \mathcal{F}(0) \mathcal{A}) \times (\forall_n \mathcal{F}(0) \mathcal{A}) \). We need functions going back and forth between the types \( \text{Id}((f_0, f_1), (g_0, g_1)) \) and \( \Pi_{R \to R_0} \mathcal{F}(1)(\mathcal{A}, R)(0, 0) \). Such functions will automatically be mutually inverse since the types in question are propositions.

Going from left to right is easy using \( \text{Id} \)-induction and \( f_1 \). To go from right to left, fix \( \phi : \Pi_{R \to R_0} \mathcal{F}(1)(\mathcal{A}, R)(0, 0) \). To show \( \text{Id}((f_0, f_1), (g_0, g_1)) \) it suffices to show \( \text{Id}(f_0, g_0) \) since the type of \( g_1 \) (or \( f_1 \)) is a proposition. By function extensionality, it suffices to show pointwise equality between \( f_0 \) and \( g_0 \). So fix \( \phi \). The only thing we can do with \( \phi \) is to apply it to \( \mathcal{E}(\mathcal{A}) \), which gives us \( \phi(\mathcal{E}(\mathcal{B})) : \mathcal{F}(1)(\mathcal{A}, \mathcal{B}) \). The relation \( \mathcal{F}(1)(\mathcal{A}, \mathcal{B}) \) is isomorphic to \( \mathcal{F}(0)(\mathcal{A}, \mathcal{B}) \) via \( \varepsilon_{\mathcal{F}}(\mathcal{A}, \mathcal{B})^{-1} \). Applying \( \varepsilon_{\mathcal{F}}(\mathcal{A}, \mathcal{B})^{-1} \) to \( (f_0, g_0) \) and \( \phi(\mathcal{E}(\mathcal{B})) \) thus gives us \( \text{Id}(\varepsilon_{\mathcal{F}}(\mathcal{A}, \mathcal{B})^{-1} f_0, \varepsilon_{\mathcal{F}}(\mathcal{A}, \mathcal{B})^{-1} g_0) \). The coherence condition on \( \varepsilon_{\mathcal{F}} \) tells us that the respective images \( \varepsilon_{\mathcal{F}}(\mathcal{A}, \mathcal{B}) \) and \( \varepsilon_{\mathcal{F}}(\mathcal{A}, \mathcal{B}) \) under the two face maps are the identity on \( \mathcal{F}(0)(\mathcal{A}, \mathcal{B}) \), and thus are \( \varepsilon_{\mathcal{F}}(\mathcal{A}, \mathcal{B}) \) and \( \varepsilon_{\mathcal{F}}(\mathcal{A}, \mathcal{B}) \). This gives \( \text{Id}(f_0, g_0) \) as desired.

Example 60 (A categorical version of Reynolds’ model, continued). On the set level, the adjoint \( \forall_n \) is defined as follows:

\[
\forall_n \mathcal{F}(0) \mathcal{A} := \left\{ f_0 : \Pi_{A \to U} \mathcal{F}(0)(\mathcal{A}, A) \land f_1 : \Pi_{R \to R_0} \mathcal{F}(1)(\mathcal{A}, R)(f_0(0), f_0(1)) \right\}
\]

The last condition says that \( f_0 \) is functorial in its argument, in the sense that if \( i \) is an isomorphism between two types \( A, B : \text{Set}_0 \), then \( f_0(A) \) and \( f_0(B) \) are suitably related via the obvious isomorphism between \( \mathcal{F}(0)(\mathcal{A}, A) \) and \( \mathcal{F}(0)(\mathcal{A}, B) \). This condition, which does not have an analogue in the set-theoretic presentation of Reynolds’ model,
is needed because we do not work with discrete domains (e.g., we use $F : M^n \to M$ rather than $F : |M|^n \to M$), as is common in other presentations of parametricity. A very similar condition does appear, e.g., in the definition of parametric limits for the category of sets in [2]. The analogous condition asserting the functoriality of $f_1$ is automatically satisfied since the codomain of $f_1$ is a proposition. On relations, we use the same definition as in Example 59.

Definition 61. A $\lambda 2$-fibration is a $\lambda^\rightarrow$-fibration $U : \mathcal{E} \to \mathcal{B}$ satisfying the following properties:

1. For each $I$ in $\mathcal{B}$, the weakening functor induced by the first projection from $I \times \Omega$ to $\Omega$ has a right adjoint $\forall_I$.

2. Beck-Chevalley: for any morphism $f : I \to J$ in $\mathcal{B}$ and object $X$ in $\mathcal{E}_J$, the canonical morphism below is an isomorphism:

$$\theta_f(f, X) : f^*(\forall_J(X)) \to \forall_I((f \times \text{id})^*(X))$$

A $\lambda 2$-fibration is split if it is a split $\lambda^\rightarrow$-fibration and the canonical morphism above is the identity.

Seely [16] essentially showed the following:

Theorem 62 (Seely). Every split $\lambda 2$-fibration $U : \mathcal{E} \to \mathcal{B}$ gives a sound model of System F in which:

- every type context $\Gamma$ is interpreted as an object $[\Gamma]$ in $\mathcal{B}$
- every type $\Gamma \vdash T$ is interpreted as an object $[\Gamma \vdash T]$ in the fiber over $[\Gamma]$.
- every term context $\Gamma ; \Delta$ is interpreted as an object $[\Gamma ; \Delta]$ in the fiber over $[\Gamma]$.
- every term $\Gamma ; \Delta \vdash t : T$ is interpreted as a morphism $[\Gamma ; \Delta \vdash t : T]$ from $[\Gamma ; \Delta]$ to $[\Gamma \vdash T]$ in the fiber over $[\Gamma]$.

A (not necessarily split) $\lambda 2$-fibration also gives a sound model of System F, due to the following:

Lemma 63. Every $\lambda 2$-fibration is equivalent to a split $\lambda 2$-fibration in a canonical way.

We now want to specify when a model of System F given by a $\lambda 2$-fibration is relationally parametric. If $\mathcal{R}$ is a cartesian closed reflexive graph category with isomorphisms, we denote by $F(\mathcal{R})$ the $\lambda^\rightarrow$-fibration induced by $\mathcal{R}$ as in Theorem 57. To formulate an abstract definition of a parametric model, we will appropriately relate a $\lambda 2$-fibration $U$ to $F(\mathcal{R})$. To see how, we revisit the simplest model, namely the System F term model. In the $\lambda 2$-fibration $U_{\text{term}}$ corresponding to the term model, the fiber over $n \in \mathbb{N}$ consists of types and terms with $n$ free type variables. Let $\mathcal{U}$ be the category consisting of closed System F types and terms between them. Then $\mathcal{U}$ induces a $\lambda^\rightarrow$-fibration, $U_{\text{set}}$, whose fiber over $n$ consists of functors $|\mathcal{U}|^n \to \mathcal{U}$ and natural transformations between them.

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A type $\alpha \vdash T$ with $n$ free variables can now be seen as functor $|\mathcal{U}|^n \to \mathcal{U}$, and a term $\alpha; x : S \vdash t : T$ as a natural transformation between $S$ and $T$. We thus have a morphism of $\lambda^\leftrightarrow$-fibrations $\mu : U_{\mathsf{term}} \to U_{\mathsf{set}}$. However, unlike $U_{\mathsf{term}}$, $U_{\mathsf{set}}$ does not admit the family of adjoints required to make it a $\lambda^2$-fibration. Still, we can view $U_{\mathsf{term}}$ as a version of $U_{\mathsf{set}}$ that “enriches” the functors and natural transformations with enough extra information to ensure that the desired adjoints exist: in this example, the information that the maps involved are not ad hoc, but come from syntax. Since these adjunctions are only applicable to non-empty contexts, no such “enrichment” should be necessary for objects and morphisms over the terminal object. And indeed, the restriction of $\mu$ to the fibers over the respective terminal objects is clearly an equivalence. These observations echo those immediately following Definition 5, and motivate our main definition:

**Definition 64.** Let $\mathcal{R}$ be a cartesian closed reflexive graph category with isomorphisms. A parametric model of System F over $\mathcal{R}$ is a $\lambda^2$-fibration $U$ together with a morphism $\mu : U \to F(\mathcal{R})$ of $\lambda^\rightarrow$-fibrations whose restriction to the fibers of $U$ and $F(\mathcal{R})$ over the terminal objects is full, faithful, and essentially surjective.

Our main theorem shows that the definition of a parametric model is indeed sensible:

**Theorem 65.** Every parametric model of System F over a cartesian closed reflexive graph category $(\mathcal{X}, (\mathcal{M}, I))$ with isomorphisms, as specified in Definition 64, is a sound model in which:

- every type $\Gamma \vdash T$ can be seen as a face map- and degeneracy-preserving reflexive graph functor $[\Gamma \vdash T] : \mathcal{M}^{[\Gamma]} \to \mathcal{M}$
- every term $\Gamma; \Delta \vdash t : T$ can be seen as a face map- and degeneracy-preserving reflexive graph natural transformation $[\Gamma; \Delta \vdash t : T] : [\Gamma \vdash \Delta] \to [\Gamma \vdash T]$, with the domain and codomain seen as reflexive graph functors into $\mathcal{X}$

**Theorem 66 (PER model).** Let $\mathcal{R}_{\mathsf{PER}}$ be the cartesian closed reflexive graph category with isomorphisms defined in Examples 7, 28, and 53. The family of adjoints defined in Example 58 makes $F(\mathcal{R}_{\mathsf{PER}})$ into a $\lambda^2$-fibration, and hence into a parametric model of System F over $\mathcal{R}_{\mathsf{PER}}$.

**Theorem 67 (Reynolds’ model).** Let $\mathcal{R}_{\mathsf{REY}}$ be the reflexive graph category with isomorphisms defined in Examples 8, 30, and 54. The family of adjoints defined in Example 59 makes $F(\mathcal{R}_{\mathsf{REY}})$ into a $\lambda^2$-fibration, and hence into a parametric model of System F over $\mathcal{R}_{\mathsf{REY}}$.

**Theorem 68 (A categorical version of Reynolds’ model).** Let $\mathcal{R}_{\mathsf{CREY}}$ be the reflexive graph category with isomorphisms defined in Examples 8, 29, and 54. The family of adjoints defined in Example 60 makes $F(\mathcal{R}_{\mathsf{CREY}})$ into a $\lambda^2$-fibration, and hence into a parametric model of System F over $\mathcal{R}_{\mathsf{CREY}}$.
We now describe a proof-relevant version of Reynolds’ model, in which witnesses of relatedness are themselves related. The construction of such a model is the subject of [12], but the development there seems to contain a major technical gap. Specifically, it is unclear how to prove the $\forall$-case in Lemma 9.4 in [12], due to the fact that when types are interpreted as discrete functors $|X|^n \to X$, the reindexing of a degeneracy-preserving functor might not be degeneracy-preserving. We already observed this in the introduction but this issue is not addressed in [12] and the proof of the lemma is not given there. Since this lemma is crucial to the soundness of the interpretation, it is unknown whether the result of [12] can be salvaged as-is. For this reason, we only reuse the main ideas of [12] for handling the higher dimensional structure and otherwise proceed independently.

Example 69. We use the same ambient category as in Example 8 and reuse the (internal) category $\text{Set}$ of types. The category $R$ of relations is almost the same as in Example 8, except that relations are now proof-relevant, i.e., $R_0 := \Sigma_{A,B:\text{Set}} A \times B \to U$. As before, we have two face maps $f_\top, f_\bot : R \to \text{Set}$ projecting out the domain and codomain of a relation and a degeneracy $\text{Eq} : \text{Set} \to R$ constructing the equality relation. Given relations $R$ on $A$ and $B$ and $S$ on $C$ and $D$, to relate two witnesses $p : R(a,b)$ and $q : S(c,d)$ we should know a priori how $a$ relates to $c$ and $b$ to $d$.

This motivates defining the category $2R$ of 2-relations, whose objects $Q$ are tuples $(Q_0^\top, Q_1^\top, Q_0^\bot, Q_1^\bot)$ of relations forming a square

\[
\begin{array}{c}
A & \longrightarrow & B \\
\downarrow & & \downarrow \\
C & \longrightarrow & D
\end{array}
\]

together with a Prop-valued predicate (also denoted $Q$) on the type of tuples of the form $((a,b,c,d), (p,q,r,s))$, where $p : Q_0^\top(a,b)$, $q : Q_1^\top(a,c)$, $r : Q_0^\bot(c,d)$, and $s : Q_1^\bot(b,d)$. This gives four face maps $f_0^\top, f_1^\top, f_0^\bot, f_1^\bot : 2R \to R$, one for each edge. We have two degeneracies from $R$ to $2R$, one replicating a relation $R$ horizontally and one vertically. More precisely, given $R$, we obtain the 2-relation $\text{Eq}_\to(R)$ by placing $R$ on top and bottom, with equality relations $\text{Eq}(R_0)$ and $\text{Eq}(R_1)$ as vertical edges, and mapping $((a,b,a,b), (p, r, r))$ to $\text{Id}(p, r)$. The symmetric version $\text{Eq}_\perp(R)$ places $R$ on left and right and assumes equality relations as horizontal edges. But we also have two other ways of turning a relation $R$ into a 2-relation: the functor $C^\top$ places $R$ on top and left, and $C^\bot(R)$ places $R$ on bottom and right, filling the remaining edges with equalities. The functors $C^\top$ and $C^\bot$ are called connections. We define terminal objects, products, exponentials, and isomorphisms in the obvious way.

Just like in Reynolds’ model, we have $f_* \circ \text{Eq} = \text{id}$. We also have further equalities:

- $f_0^* \circ \text{Eq} = \text{id}$
- $f_1^* \circ \text{Eq} = \text{Eq} \circ f_*$ for a fixed $* \in \text{Bool}$
\[ \bullet f_{\lambda} \circ \mathsf{Eq} = \mathsf{id} \]
\[ \bullet f_0 \circ \mathsf{Eq} = \mathsf{Eq} \circ f_\ast \quad \text{for a fixed } \ast \in \mathsf{Bool} \]
\[ \bullet f_\ast \circ C_* = \mathsf{id} \quad \text{for } l \in \{0, 1\} \text{ and a fixed } \ast \in \mathsf{Bool} \]
\[ \bullet f_l \circ C_* = \mathsf{Eq} \circ f_l \quad \text{for } l \in \{0, 1\} \text{ and a fixed } \ast \in \mathsf{Bool} \]

Moreover, the compositions \( \mathsf{Eq} \circ \mathsf{Eq} \circ \mathsf{Eq} \circ \mathsf{C}_\top \circ \mathsf{Eq} \circ \mathsf{C}_\bot \circ \mathsf{Eq} \) are all naturally isomorphic.

The structure described above induces two \( \lambda^\top \)-fibrations of interest: the first one is induced by combining the first two levels, the categories \( \mathsf{Set} \) and \( \mathsf{R} \), into a cartesian closed reflexive graph category with isomorphisms \( \mathsf{R}_{\mathsf{PREY}} \); this is the fibration \( \mathcal{F}(\mathsf{R}_{\mathsf{PREY}}) \). We recall that the objects of \( \mathcal{F}(\mathsf{R}_{\mathsf{PREY}}) \) over \( n \) are pairs \( \{\mathcal{F}(l) : \mathcal{M}(l)^n \to \mathcal{M}(l)\}_{l \in \{0,1\}} \) of functors that commute with the two face maps from \( \mathsf{R} \) to \( \mathsf{Set} \) on the nose, as well as with the degeneracy \( \mathsf{Eq} \) up to a suitably coherent natural isomorphism \( \varepsilon_\mathcal{F} \). The morphisms are pairs \( \{\eta(l) : \mathcal{F}(l) \to \mathcal{G}(l)\}_{l \in \{0,1\}} \) of natural transformations that respect both face maps from \( \mathsf{R} \) to \( \mathsf{Set} \) and the degeneracy \( \mathsf{Eq} \).

The second fibration, which we call \( \mathcal{F}_{\mathsf{2D}} \), is induced in much the same way, but taking into account all three levels. This means that the objects over \( n \) are triples \( \{\mathcal{F}(l) : \mathcal{M}(l)^n \to \mathcal{M}(l)\}_{l \in \{0,1,2\}} \) of functors that commute with all face maps – the two from \( \mathsf{R} \) to \( \mathsf{Set} \) as well as the four from \( \mathsf{2R} \) to \( \mathsf{R} \) – on the nose and all degeneracies \( \mathsf{Eq}, \mathsf{Eq}_\top, \mathsf{Eq}_\bot \) and connections \( \mathsf{C}_\top, \mathsf{C}_\bot \) up to suitably coherent natural isomorphisms. Here “suitably coherent” means taking into account not only the equality \( f_\ast \circ \mathsf{Eq} = \mathsf{id} \) but the additional equalities involving \( \mathsf{Eq} \circ \mathsf{Eq}_\top, \mathsf{C}_\top, \mathsf{C}_\bot \) as well. For example, the image of the isomorphism witnessing the commutativity of \( \mathcal{F} \) with \( \mathsf{Eq}_\top \) under the face map \( f_\lambda \) must be precisely \( \varepsilon_\mathcal{F} \circ f_\lambda \). Analogously, the morphisms are triples \( \{\eta(l) : \mathcal{F}(l) \to \mathcal{G}(l)\}_{l \in \{0,1,2\}} \) of natural transformations that respect all face maps, degeneracies, and connections. We have the obvious forgetful morphism of \( \lambda^\top \)-fibrations from \( \mathcal{F}_{\mathsf{2D}} \) to \( \mathcal{F}(\mathsf{R}_{\mathsf{PREY}}) \) that only retains the structure pertaining to levels \( 0 \) and \( 1 \).

The fibration \( \mathcal{F}_{\mathsf{2D}} \) admits a family of adjoints to weakening functors as follows. The adjoint \( \forall_n \mathcal{F}(0) A \) is the type

\[
\begin{align*}
\{ f_0 : \Pi_{A: \mathsf{Set}} \mathcal{F}(0)(A, A) & \} \\
\{ f_1 : \Pi_{R: \mathsf{R}_0} \mathcal{F}(1)(\mathsf{Eq}(A), R) (f_0(R_d), f_0(R_d)) & \} \\
\{ f_2 : \Pi_{Q: \mathsf{2R}_0} \mathcal{F}(2)(\mathsf{Eq}(\mathsf{Eq}(A)), Q) (f_0 Q^0, f_0 Q^1, f_0 Q^0, f_0 Q^0, f_1 Q^0, f_1 Q^0, f_1 Q^0) & \} \\
\{ i_{\mathsf{M}(0)}, \mathcal{F}(0)(\mathsf{id}_{\mathsf{M}(0)}(A), i) f_0(i_\varepsilon) = f_0(i_d) & \} \\
\{ i_{\mathsf{M}(1)}, \mathcal{F}(1)(\mathsf{id}_{\mathsf{M}(1)}(A), i) (f_0(i_d), f_0(i_d)) f_1(i_\varepsilon) = f_1(i_\varepsilon) & \}
\end{align*}
\]

In the type of \( f_2 \), we could have just as well used any of the other functors \( \mathsf{Eq}_\top, \mathsf{C}_\top, \mathsf{C}_\bot \) instead of \( \mathsf{Eq}_\top \) since as observed above, their compositions with \( \mathsf{Eq} \) are all naturally isomorphic. We next define \( \forall_n \mathcal{F}(1) \mathcal{R} \) to be the relation with domain \( \forall_n \mathcal{F}(0) \mathcal{R}_d \).
and codomain \( \forall_n F(0) \overline{R_c} \) mapping \(((f_0, f_1, f_2), (g_0, g_1, g_2))\) to

\[
\begin{align*}
\{ \phi &: \Pi_{R_c} R \rightarrow (1) (R, R) (f_0(R_a), g_0(R_c)) \} \\
\phi_{Eq} &: \Pi_{Q:2R_c} F(2) (\overline{E_{Eq}}(R), Q) ((f_0 Q^0_T, f_0, Q^0_T)^T, g_0 Q^0_T, g_0 Q^0_T), \\
& (f_1 Q^0_T, f_1 Q^0_T, g_1 Q^0_T, g_1 Q^0_T)) & & (1) \\
\phi_{Eq} &: \Pi_{Q:2R_c} F(2) (\overline{E_{Eq}}(R), Q) ((f_0 Q^0_T, f_0, Q^0_T)^T, g_0 Q^0_T, g_0 Q^0_T), \\
& (f_1, f_1 Q^0_T, g_1 Q^0_T, g_1 Q^0_T)) & & (2) \\
\phi_{C_\parallel} &: \Pi_{Q:2R_c} F(2) (\overline{C_\parallel(R)}, Q) ((f_0 Q^0_T, f_0, Q^0_T)^T, g_0 Q^0_T, g_0 Q^0_T), \\
& (f_1 Q^0_T, f_1 Q^0_T, g_1 Q^0_T, g_1 Q^0_T)) & & (3) \\
\phi_{C_\parallel} &: \Pi_{Q:2R_c} F(2) (\overline{C_\parallel(R)}, Q) ((f_0 Q^0_T, f_0, Q^0_T)^T, g_0 Q^0_T, g_0 Q^0_T), \\
& (f_1 Q^0_T, f_1 Q^0_T, g_1 Q^0_T, g_1 Q^0_T)) & & (4) \\
\Pi_{i:M(1)} F(1) (\overline{d_{M(1)}(R)}), i \} (f_0(i_id)_{id}^0, g_0(i_id)_{id}^0) \phi_{i}(i_i) = \phi_{i}(1_i) \\
\end{align*}
\]

The component \( \phi_{Eq} \) asserts that \( \phi \) appropriately interacts with the degeneracy \( \overline{E_{Eq}} \) and similarly for the analogous components \( \phi_{Eq} \), \( \phi_{C_\parallel} \), \( \phi_{C_\parallel} \). We define \( \forall_n F(2) Q \) to be the 2-relation with underlying tuple of relations

\[
(\forall_n F(1) Q^0_T, \forall_n F(1) Q^1_T, \forall_n F(1) Q^0_1, \forall_n F(1) Q^1_1)
\]

mapping \(((f_0, f_1, f_2), (g_0, g_1, g_2), (h_0, h_1, h_2), (l_0, l_1, l_2)), ((\phi_0, \ldots), (\phi_1, \ldots), (\phi_2, \ldots), (\phi_3, \ldots)))\) to the proposition

\[
\Pi_{Q:2R_c} F(2) (\overline{Q}, Q) ((f_0 Q^0_T, f_0 Q^0_T)^T, h_0 Q^0_1, l_0 Q^0_1), \\
(\phi_0 Q^0_T, \phi_1 Q^1_T, \phi_2 Q^0_1, \phi_3 Q^1_1))
\]

Finally, unlike the frameworks \([2, 4, 5, 7, 10, 15]\), our definition of a parametric model recognizes the above proof-relevant model:

**Theorem 70** (Proof-relevant model). The family of adjoints defined in Example 69 makes \( F_{\text{adj}} \) into a \( \lambda2 \)-fibration, and hence into a parametric model of System \( F \) over \( R_{\text{PREY}} \).

**Proof sketch.** Faithfulness follows because having \( \eta(0), \eta(1) \) fixed, there is a unique way to define \( \eta(2) \): since \( \eta \) has to respect the degeneracy \( \overline{E_{Eq}} \), we must have

\[
\eta(2) \circ \overline{R_c} \circ \overline{E_{Eq}} = \overline{E_{Eq} \circ \overline{R_c} \circ \overline{E_{Eq}}} \eta(1)
\]

where \( \overline{E_{Eq}} \), \( \overline{E_{Eq}} \) are the natural isomorphisms witnessing the fact that \( F, G \) by assumption preserve \( \overline{E_{Eq}} \) (we could have used any of the other functors \( \overline{E_{Eq}}, C_T, C_\parallel \) as well). This gives at most one possible value for \( \eta(2) \). Fullness follows since the triple \( \{ \eta(l) \}_{l \in \{0, 1, 2\}} \) with \( \eta(2) \) as given above indeed respects all face maps, degeneracies, and connections (in fact it is only necessary to check the respecting of face maps since the predicates at level 2 are proof-irrelevant). Finally, essential surjectivity follows from the fact that the reflexive graph functor \( (\overline{F}(0), \overline{F}(1)) \) is isomorphic to the reflexive graph functor \( (\overline{F}(0), \overline{F}(0)) \) via the reflexive graph natural transformation
(id, ε₇) (the fact that this transformation is face-map preserving again uses the coherence \(\mathcal{R}(f) \circ ε₇ = id\)). But \((\mathcal{F}(0), Eq(\mathcal{F}(0)))\) clearly belongs to the image since it can be extended e.g., to the triple \((\mathcal{F}(0), Eq(\mathcal{F}(0)), Eq(\mathcal{F}(0)))\). □

7 Discussion

We can now be more specific about how our approach compares to the external approaches in [4, 7, 10, 15], all of which are based on a reflexive graph of \(\lambda_2\)-fibrations. The definition in [4] appears to be too restrictive: it requires a comprehension structure that, e.g., the \(\lambda_2\)-fibration corresponding to Reynolds’ model does not admit. In addition, none of these frameworks seem to recognize the \(\lambda\)-fibration corresponding to the proof-relevant model as parametric, for the following reason: it is unclear how to define the family of adjoints for the second fibration (called \(r\) in [7]) of “heterogeneous” reflexive graph functors in a way that is compatible with the adjoint structure on the original \(\lambda_2\)-fibration. This is because unlike in the proof-irrelevant case, the definition of \(\forall_n \mathcal{F}(1)\) now has conditions such as the one witnessed by \(\phiₙ\) which are only meaningful for “homogeneous” reflexive graph functors, i.e., those where the domain and codomain of \(\mathcal{F}(1)(\mathcal{R})\) are given by the same functor \(\mathcal{F}(1)\), albeit applied to different arguments (\(\mathcal{R}_a\) vs. \(\mathcal{R}_c\)). Our definition does not rely on or require two compatible adjoint structures, which is why we are indeed able to recognize the proof-relevant model as parametric.

We indicate three directions for future work. Readers interested in applications of parametricity will notice that we do not require conditions such as \(\text{op-cartesianness}\) or \(\text{fullness of certain maps or well-pointedness}\) of certain categories. This follows the spirit of [7], where the notion of parametricity pertains to the suitable interaction with (what we call) face maps and degeneracies. Specific applications such as establishing the Graph Lemma and the existence of initial algebras are left for another occasion. Readers fond of type theory might wonder about possible models expressed in the intensional version of dependent type theory. Although currently there are no well-known models for which the latter would be the right choice of meta-theory, that might change with more research into higher notions of parametricity. Finally, readers familiar with cubical sets no doubt recognized the structure of sets, relations, and 2-relations with face maps, degeneracies, and connections from the last section as the first few levels of the cubical hierarchy, and wonder whether one can formulate the analogous notion of 2, 3, \ldots-parametricity using this hierarchy. We conjecture the answer to be a YES! and plan to pursue this question in future work.

Acknowledgments This research is supported by NSF awards 1420175 and 1545197. We thank Steve Awodey and Peter Dybjer for helpful discussions. We also thank the anonymous referee who independently suggested the formulation of Reynolds’ model in Example 30, which appeared in our earlier preprint [8].
References


