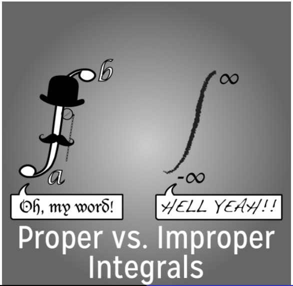
7.6 Improper Integrals (Infinity and Beyond)

- If you see an integral with ∞ in it, or infinite discontinuities
- Express the integral as a proper one via limit (or limits) to any problem(s), like $\int_a^{\infty} f(x)\, dx = \lim_{b \to \infty} \int_a^b f(x)\, dx$
- Integrate and evaluate the limit
- The integral converges to a finite number if the limit(s) exist, and diverges otherwise

What I want you to show me... The above steps.



Clicker Question

1. Which integral is improper?

a) $\int_0^1 e^{-x}\, dx$

b) $\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \sin(x)(\cos(x))^{\frac{-1}{2}}\, dx$

c) $\int_0^1 \arcsin(x)\, dx$

d) more than one of the above

e) none of the above

Clicker Question

2. What is a useful method for $\displaystyle\int_0^{0.5} \frac{1}{x^2\sqrt{1-x^2}}\,dx$?

a) Integration by improper & w-substitution
b) Integration by improper & parts
c) Integration by improper & partial fractions
d) Integration by improper & trigonometric substitution
e) More than one of the above

Clicker Question

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e) More than one of the above

$$\int_0^{\frac{1}{2}} \frac{1}{x^2\sqrt{1-x^2}}\,dx = \lim_{a\to 0^+} \int_{x=a}^{x=\frac{1}{2}} \frac{1}{x^2\sqrt{1-x^2}}\,dx.$$

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$\int_0^{\frac{1}{2}} \frac{1}{x^2\sqrt{1-x^2}}\,dx = \lim_{a\to 0^+} \int_{x=a}^{x=\frac{1}{2}} \frac{1}{x^2\sqrt{1-x^2}}\,dx$. Let $x=\sin(\theta)$.

Then $dx=\cos(\theta)d\theta$. $\int = \cdots = \lim_{a\to 0^+} \int_{x=a}^{x=\frac{1}{2}} \frac{\cos(\theta)d\theta}{\sin(\theta)^2\cos(\theta)}$

Clicker Question

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Then $dx = \cos(\theta)d\theta$. $\int = \cdots = \lim_{a\to 0^+} \int_{x=a}^{x=\frac{1}{2}} \frac{\cos(\theta)d\theta}{\sin(\theta)^2\cos(\theta)} =$

$\lim_{a\to 0^+} \int_{x=a}^{x=\frac{1}{2}} \csc^2(\theta)d\theta =$

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$\lim_{a\to 0^+} \int_{x=a}^{x=\frac{1}{2}} \csc^2(\theta)d\theta = \lim_{a\to 0^+} -\cot(\theta)\Big|_{x=a}^{x=\frac{1}{2}} =$

$\lim_{a\to 0^+} -\cot(\theta)\Big|_{\theta=\arcsin(a)}^{\theta=\arcsin(\frac{1}{2})} = \lim_{a\to 0^+} \cot(\arcsin\frac{1}{2}) - \cot(\arcsin(a))$

History and Applications

- 1821 monograph, Augustin-Louis Cauchy put forward a definition of integral that is directly based on the interpretation of area under graph of function and had limits. Before it was antideriv at endpoints.
- 1854 thesis, Bernhard Riemann investigated when an unbounded function can still be integrable
- all over mathematics, physics, computer science, economics, statistics, engineering, etc
- unbounded: integer programing, compound continuously, density functions and expectations of continuous random variables...
- point singularity: Cauchy principal value in potential theory and harmonic analysis...

Improper and What Method?

$$\int_0^{\infty} \frac{1}{9x+4}\,dx$$

Improper and What Method?

$$\int_0^{\infty} \frac{1}{9x+4}dx$$ w-subs

$$\int_0^1 4\ln(x)dx$$

Improper and What Method?

$$\int_0^{\infty} \frac{1}{9x+4}dx$$ w-subs

$$\int_0^{1} 4\ln(x)dx$$ parts

$$\int_0^{\infty} xe^{-x^2}dx$$

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$\int_0^{\infty} \frac{4x}{e^x}dx$

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$$\int_0^{\infty} \frac{4x}{e^x}dx$$ parts

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{\cos(x)}}dx$$

Improper and What Method?

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$\int_0^{1} 4\ln(x)dx$ parts

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$\int_0^{\infty} \frac{4x}{e^x}dx$ parts

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{\cos(x)}}dx$ w-subs

$\int_0^{7} \frac{-1}{x^2-49}dx$

Improper and What Method?

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$$\int_0^{1} 4\ln(x)dx$$ parts

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$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{\cos(x)}}dx$$ w-subs

$$\int_0^{7} \frac{-1}{x^2-49}dx$$ partial fractions

$$\int_0^{1} \frac{\ln(x)}{2x}dx$$

Improper and What Method?

$\int_0^{\infty} \frac{1}{9x+4}dx$ w-subs

$\int_0^1 4\ln(x)dx$ parts

$\int_0^{\infty} xe^{-x^2}dx$ w-subs

$\int_0^{\infty} \frac{4x}{e^x}dx$ parts

$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{\cos(x)}}dx$ w-subs

$\int_0^7 \frac{-1}{x^2-49}dx$ partial fractions

$\int_0^1 \frac{\ln(x)}{2x}dx$ w-subs

$\int_0^5 \frac{1}{\sqrt{25-x^2}}dx$

Improper and What Method?

$$\int_0^{\infty} \frac{1}{9x+4}dx$$ w-subs

$$\int_0^{1} 4\ln(x)dx$$ parts

$$\int_0^{\infty} xe^{-x^2}dx$$ w-subs

$$\int_0^{\infty} \frac{4x}{e^x}dx$$ parts

$$\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} \frac{\sin(x)}{\sqrt{\cos(x)}}dx$$ w-subs

$$\int_0^{7} \frac{-1}{x^2-49}dx$$ partial fractions

$$\int_0^{1} \frac{\ln(x)}{2x}dx$$ w-subs

$$\int_0^{5} \frac{1}{\sqrt{25-x^2}}dx$$ trig sub

$$\int_1^{\infty} \frac{1}{x^2+1}dx$$

Improper and What Method?

$\int_0^{\infty} \frac{1}{9x+4}dx$ w-subs

$\int_0^{1} 4\ln(x)dx$ parts

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$\int_0^{5} \frac{1}{\sqrt{25-x^2}}dx$ trig sub

$\int_1^{\infty} \frac{1}{x^2+1}dx$ calc 1–on sheet (trig sub works too)