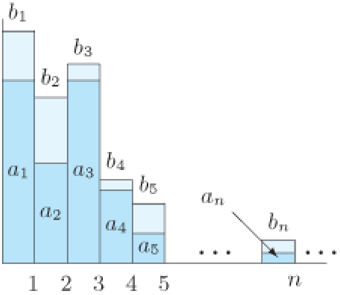
9.4: (Direct) Comparison Test

- For positive terms $a_n \leq b_n$
- If a series $\sum b_n$ converges, then so does any smaller (term by term) series
- If a series $\sum a_n$ diverges, then so does any larger (term by term) series
- Example: $\sum \frac{1}{ln(n!)}$ $\qquad \frac{1}{ln(n!)} > \frac{1}{ln(n^n)} = \frac{1}{nln(n)}$



Picture credit: *Calculus: Single Variable*

- It can be very difficult to identify $a_n \leq b_n$ for the direct comparison test—we can develop easier to use and more powerful tests.
- Examine limits of ratios when a series is not geometric in the Limit Comparison Test and the Ratio Test

Picture credit: `RickLantona.com`

9.4: Limit Comparison Test

- If $a_n > 0$ and $b_n > 0$ eventually, and $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then $\sum a_n$ behaves the same way as $\sum b_n$.

WHERE IS THE LIMIT?®

Image: http://notengofuerzaspararendirme.com/las-cosas-por-su-nombre/where_is_the_limit_petit/

- Polynomial quotients: $\sum \frac{p(n)}{q(n)}$: try $b_n =$ ratio highest powers
- Example: $\sum \frac{n^2-4}{4n^3+3n-8}$

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  Compare to $\sum \frac{n^2}{4n^3} = \sum \frac{1}{4n}$ $b_n > 0$

$\lim_{n\to\infty} \frac{a_n}{b_n} =$

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$$\lim_{n\to\infty} \frac{a_n}{b_n} = \ldots = \lim_{n\to\infty} \frac{1-\frac{4}{n^2}}{1+\frac{3}{4n^2}-\frac{8}{4n^3}} =$$

9.4: Limit Comparison Test

- If $a_n > 0$ and $b_n > 0$ eventually, and $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$, then $\sum a_n$ behaves the same way as $\sum b_n$.

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- Polynomial quotients: $\sum \frac{p(n)}{q(n)}$: try $b_n =$ ratio highest powers
- Example: $\sum \frac{n^2-4}{4n^3+3n-8}$ assumption: $a_n > 0$ eventually

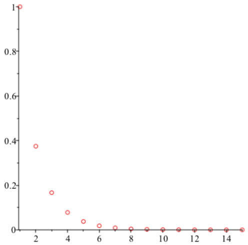
Compare to $\sum \frac{n^2}{4n^3} = \sum \frac{1}{4n}$ $b_n > 0$

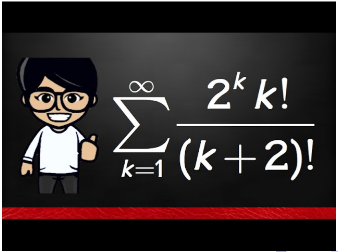
$$\lim_{n\to\infty} \frac{a_n}{b_n} = \ldots = \lim_{n\to\infty} \frac{1-\frac{4}{n^2}}{1+\frac{3}{4n^2}-\frac{8}{4n^3}} = 1 \qquad 0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$$

Original series diverges as $\sum \frac{1}{4n}$ diverges by integral test.

9.4: Ratio Test: Is the Series Approximately Geometric?

- For $\sum a_n$, if $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$
  $L < 1$ implies convergence
  $L > 1$ implies divergence
  $L = 1$ gives no information
- useful: algebra often reduces factorials (n!) or exponents
- Though not geometric, if the ratio in the limit is manageable and approaches $L < 1$ then the series converges





Picture credit: `https://i.ytimg.com/vi/p7qf_rPvLVs/sddefault.jpg` when the terms of a series do go to 0 in the terms $\not\to 0$

or

the ratio test yields 1

when the terms of a series do go to 0 in the terms $\not\to 0$

or

the ratio test yields 1

inconclusive, so try another test

Image: www.clipartkid.com/images/813/psd-detail-sad-face-official-psds-L4WZJ1-clipart.png

Clicker Question

1. Which tests will help you determine if the series $\sum_{n=1}^{\infty} e^{-n}$ converges or diverges? Examine assumptions and conclusions on the Series Theorems handout.

a) geometric series
b) integral test
c) ratio test
d) more than one of the above but not all
e) all of a, b and c

After you have responded to the clicker question, select one test to fully document

Clicker Question

2. Can we use the limit comparison test on $\sum_{n=1}^{\infty} \frac{5n+3}{6n^2}$

a) yes and I have a good reason why
b) yes but I am unsure of why
c) no, but I am unsure of why
d) no, and I have a good reason why
e) I was unable to use the limit comparison test

Test	useful	converges if	diverges if
Limit Comp	polynomial quotient a_n $a_n, b_n > 0$ eventually $0 < \lim_{n\to\infty} \frac{a_n}{b_n} < \infty$	$\sum b_n$ conv	$\sum b_n$ div

Clicker Question

3. For $\sum_{n=0}^{\infty} 2^{-n}$ compute $\lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = L$ as in the ratio test

a) $L < 1$ so the ratio test shows the series converges

b) $L > 1$ so the ratio test shows the series diverges

c) $L = 1$ so the ratio test is inconclusive

d) L does not exist so the ratio test is inconclusive

Next, what are all the series theorems we can successfully apply here?

9.4: Alternating Series Test

- Decreasing terms, alternating terms
- If $|a_n| \geq |a_{n+1}|$ and $\lim_{n\to\infty} |a_n| = 0$, then the alternating series converges.

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \frac{1}{1} - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \dots$$

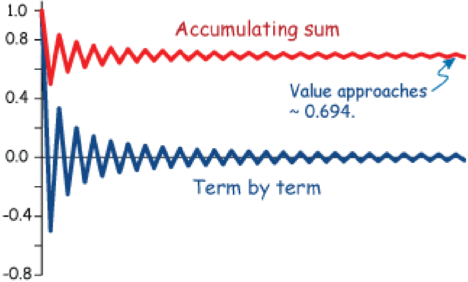


Image: http://www.drcruzan.com/Images/AlternatingSeries/AltSeriesExampleGraph.png

Clicker Question

4. Which are valid arguments?

a) $\sum \frac{4}{n}$ converges because $\lim_{n\to\infty} \frac{4}{n} = 0$

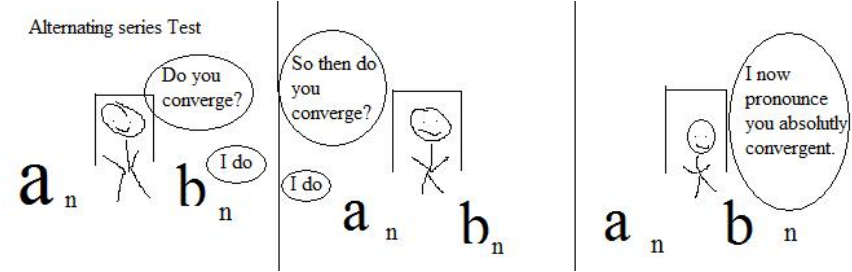
b) $\sum (-1)^n n^2$ converges by the alternating series test

c) both are valid

d) neither are valid

Test	useful	converges if	diverges if
Terms $\not\to 0$	$\sum a_n$, $a_n \not\to 0$	inconclusive	$a_n \not\to 0$
Alternating Series Test	alternating terms $\lvert a_n \rvert$ decreasing	$\lim_{n\to\infty} \lvert a_n \rvert = 0$	

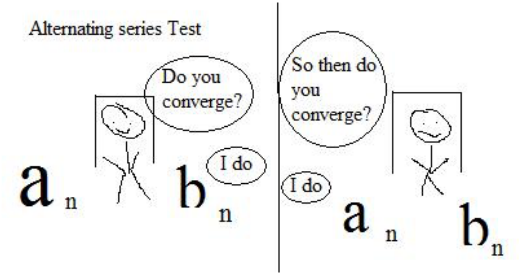
9.4: Absolute or Conditionally Convergent?



$$a_n = |b_n|$$

Image http://math.cos.ucf.edu/~/anevai/courses/creative-works/mac2312-a/creative-3/resources/thomas.jpg

9.4: Absolute or Conditionally Convergent?



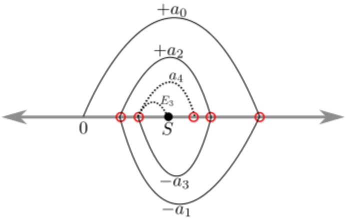
$$a_n = |b_n|$$

Image http://math.cos.ucf.edu/~/anevai/courses/creative-works/mac2312-a/creative-3/resources/thomas.jpg

Conditionally Convergent: $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$

Alternating Series Estimation

- Truncation error of using S_n, the partial sum, to approximate an alternating series of $|a_n|$ decreasing terms, is less than the absolute value of next term in the series $|a_{n+1}|$



Alternating Series Estimation

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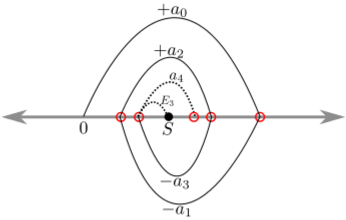
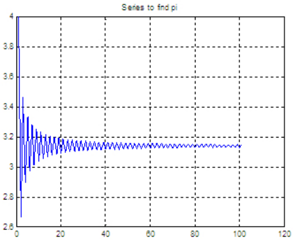




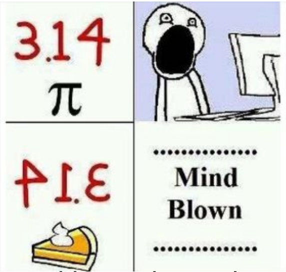
- Example $\sum_{n=1}^{\infty} (-1)^{n+1} \frac{1}{n^2}$
  S_9 has an error bounded by $|a_{10}| = |(-1)^{10+1} \frac{1}{(10)^2}| = \frac{1}{(10)^2}$

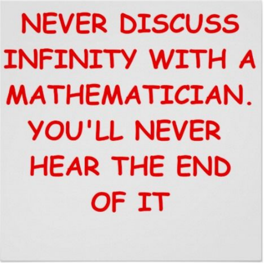
Images: http://calculus.seas.upenn.edu/?n=Main.AlternatingSeries,

http://www.matrixlab-examples.com/harmonic-series.html

History

- ratio test: first published by Jean le Rond d'Alembert and is sometimes known as d'Alembert's or Cauchy test
- alternating series test: used by Gottfried Leibniz also known as Leibniz's test, Leibniz's rule, or criterion





- widely used in mathematics and other quantitative disciplines such as physics, computer science, & finance.

Images `https://i.kinja-img.com/gawker-media/image/upload/s--x6az_AUE--/c_fit,fl_progressive,q_80,w_320/19crwoe2xg55mjpg.jpg`, `https://s-media-cache-ak0.pinimg.com/736x/ab/b0/7a/abb07a14c8dcc424d9a9a0676640c7b8.jpg`