10.3: Estimating the Unknown and 10.4 Error Bounds

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- $\arctan 1 = \frac{\pi}{4}$, so Gottfried Leibniz formula for π
  $\frac{\pi}{4} = arctan1 = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \ldots = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1}$
- Converges extremely slowly. Want 10 decimal places? Need 5 billion terms.
- Convergence acceleration
- Approximate values of functions and integrals which are not elementary

10.3: New Series from Old

- *The DNAs of two people are chemically combined to form the DNA of their offspring, which results in specific phenotypic traits being inherited by the offspring from its parents.*
- *The power series expansions of functions are mathematically combined to form the power series expansion of their "offspring" (through compositions, combinations, derivatives, antiderivatives, etc.), and specific mathematical traits (such as the series' radius of convergence) are inherited by the offspring from its "parents".*
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  [R. Travis Kowalski]

Q: What did Taylor's preteen daughter say when he encouraged her to be unique?

A: "Stop trying to differentiate me, Dad!" [Sarah Zureick-Brown]

Clicker Question

1. What is the *linear* term of the Taylor series of $t\sin(2t)$ at $t = 0$?
(Hint: sub $2t$ into Taylor series for $\sin x$ and multiply by t).

a) 0
b) t
c) $2t$
d) $2t^2$
e) $\sin(2t)$

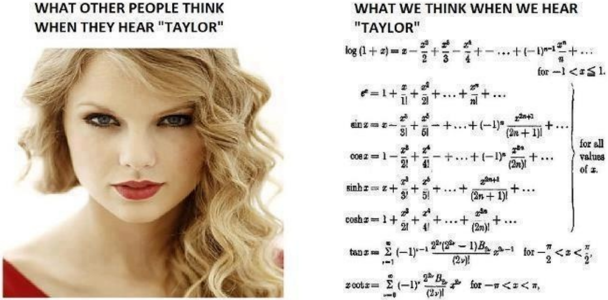

Image: `https://www.mathfunny.com/tag/taylor/`

Clicker Question

2. Suppose we find a Taylor series for the function $f(x)$ about 5. Where would we expect this Taylor series to probably give us the best estimate?

a) $x = 0$
b) $x = 3$
c) $x = \pi$
d) $x = 8$
e) no way to tell without more information

Q: Why do Taylor polynomials fit the original function so well near the center?

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Q: Why do Taylor polynomials fit the original function so well near the center?
A: Because they are "Taylor" made.

10.4: Graphical and Algebraic Error Bounds

We can use the largest $y-$value difference between a function and a Taylor polynomial on an interval to bound the error.

Bound the error of using $P_1(x) = x$ for $\sin x$ on [0,2]?

using $P_3(x) = x - \frac{x^3}{3!}$?

using $P_5(x) = x - \frac{x^3}{3!} + \frac{x^5}{5!}$?

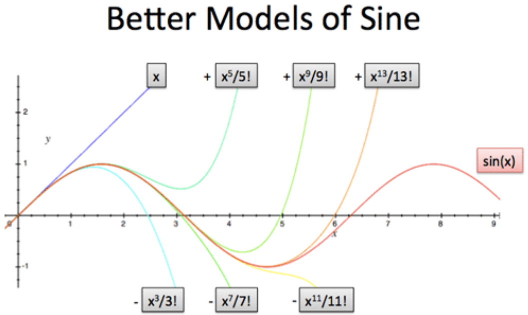

Image: http://betterexplained.com/wp-content/uploads/sine/sine-better-models.png

Lagrange Error Bound/Taylor's Inequality

If f and all its derivatives are continuous, then the error between $f(x)$ and $P_n(x)$, the Taylor polynomial about x=a, is bounded by

$$|f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$$

if $|f^{(n+1)}| \leq M$ between a and x

Forensic mathematics analogy: *specificity/sensitivity test for DNA, which is a precise measure of how well partial genetic information can accurately represent a phenotypic picture of an individual.* [R. Travis Kowalski]

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Example: Find an error bound when we approximate $\cos(.1)$ by the degree 3 Taylor polynomial for $f(x) = \cos x$ about $x = 0$.

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$$|f(.1) - P_3(.1)| \leq \frac{M}{(3+1)!}|.1-0|^{3+1}$$

Lagrange Error Bound/Taylor's Inequality

If f and all its derivatives are continuous, then the error between $f(x)$ and $P_n(x)$, the Taylor polynomial about a, is bounded by

$|f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$, with $|f^{(n+1)}| \leq M$ for a to x

Example: Find an error bound when we approximate $\cos(x)$ for values of x close to 0 at $x = b$ by the degree 6 Taylor polynomial about $x = 0$.

$$|f(b) - P_6(b)| \leq \frac{M}{(6+1)!}|b-0|^{6+1}$$

Exact at b=0
Error at b=0.5?
Error at b=0.1?
Sometimes we can use the bound to confirm whether the Taylor series converges to the function!

Tips for Finding $|f^{(n+1)}| \leq M$ on an interval

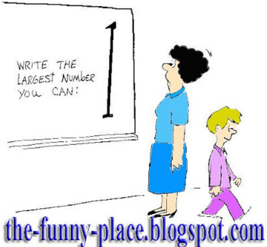

Image: http://the-funny-place.blogspot.com/2008/08/

- increasing functions, like $|f^{(n+1)}| = e^x$ or $\ln x$, achieve their maximum at the right most endpoint, decreasing at the left
- test out both endpoints in $|f^{(n+1)}|$ and choose the larger

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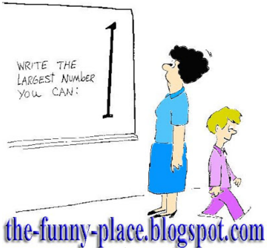

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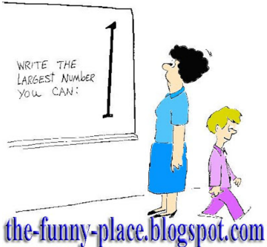
- $|f^{(n+1)}| = \sin x$ or $\cos x$ are bounded by 1

  If we are in a smaller interval where the function is only increasing or decreasing, we can get a better M.

  Ex: on $[0, \frac{\pi}{4}]$ $f^{(n+1)} = e^{\sin x}$

Tips for Finding $|f^{(n+1)}| \leq M$ *on an interval*



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Ex: on $[0, \frac{\pi}{4}]$ $f^{(n+1)} = e^{\sin x}$ achieves it's maximum at $e^{\sin \frac{\pi}{4}}$

Clicker

Use the Lagrange Error Bound/Taylor's Inequality to find the maximum possible error in estimating e^{-2} using the Taylor polynomial of degree 8 about 0 for $f(x) = e^x$

(Hint: compute and bound $|f^{(n+1)}|$ on [-2,0] and fit that M into: $|f(x) - P_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1}$, with $|f^{(n+1)}| \leq M$ for a to x)

a) 0
b) $\frac{2^9}{9!}$
c) $\frac{e^{-2}}{9!}2^9$
d) ∞
e) other

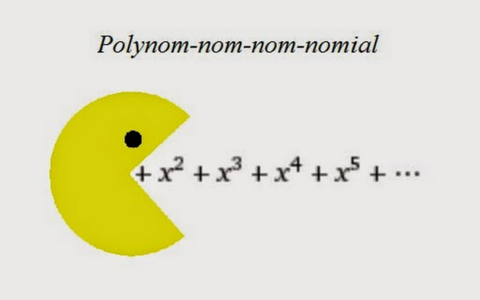

Image: `https://ifunny.co/fun/WyaUCS5P3?gallery=tag&query=pacman`

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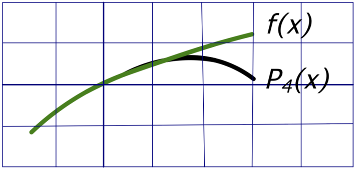
Using only the graph, what is a good bound on the largest error of using the 4th degree Taylor polynomial on the interval where they are graphed. Each tick mark is 1 unit.



a) approximately .1
b) approximately 1
c) approximately 4.5
d) have to take the integral to get the area under the curve
e) none of the above

Clicker

Use the Lagrange Error Bound/Taylor's Inequality to bound the error of using the linear approximation of $\arctan x$ about 0 to estimate $\arctan \frac{1}{2}$.

a) 0

b) $\frac{1}{1!}(\frac{1}{2})^1 = .5$

c) $\frac{1}{2!}(\frac{1}{2})^2 = .125$

d) $\frac{\frac{2\frac{1}{2}}{(1+\frac{1}{4})^2}}{2!}(\frac{1}{2})^2 = .08$

e) other

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Taylor Series—Big Picture

- What are Taylor series important for? List two ideas.

Taylor Series—Big Picture

- What are Taylor series important for? List two ideas.
- What are you left wondering about Taylor series or have a question on?

Orbifold Series (DGW-)

Asymptotic expansion of the heat trace:

$$
\begin{aligned}
&[\frac{SurfaceArea(\mathcal{O})}{4\pi}]\frac{1}{t} \\
&+ [\frac{1}{64\sqrt{\pi}} \int_{MirrorLocus(\mathcal{O})} \tau]\sqrt{t} \\
&+ \frac{\chi(\mathcal{O})}{6} + \sum_{i=1}^{k} \frac{1}{m_i}\frac{m_i^2-1}{12} + \sum_{j=1}^{r} \frac{1}{2n_j}\frac{n_j^2-1}{12} \\
&+ [\frac{a_2}{4\pi} + \sum_{i=1}^{k} \frac{R_{1212}(m_i^4+10m_i^2-11)}{360m_i} + \sum_{i=1}^{r} \frac{R_{1212}(n_i^4+10n_i^2-11)}{720n_i}]t \\
&+ [\frac{length(MirrorLocus(\mathcal{O}))}{8\sqrt{\pi}}]\frac{1}{\sqrt{t}} \\
&+ \dots
\end{aligned}
$$