7.6 Improper Integrals Introduction

$\int_a^b f(x)dx$ is called improper if there is

an infinite interval of integration ($a = -\infty$ and/or $b = \infty$)

OR an infinite discontinuity ($f(x)$ has a vertical asymptote on the region of integration)

1. Let's examine $\int_0^\infty e^{-.4x}\,dx$.
   (a) **Sketch a rough plot** from $x = 0$. Then **shade in the area** that represents the integral from 0 to ∞.

   (b) The integral is improper because it has an infinite interval of integration ($b = \infty$). Improper integrals are handled as limits:

   $$\lim_{b\to\infty} \int_0^b e^{-.4x}\,dx$$

   As in [1], the integral to b could represent the fraction of light bulbs from a given company that fail within the first b months, while the integral to ∞ could represent the fraction of light bulbs that fail *eventually*, so "it is natural to consider questions where we desire to integrate over an interval whose upper limit grows without bound."

   We continue by integrating as usual. **What method of integration did we use here?**

   $$= \lim_{b\to\infty} \left.\frac{e^{-.4x}}{-.4}\right|_0^b$$

   (c) Next we plug in the endpoints:

   $$\lim_{b\to\infty} \frac{e^{-.4b}}{-.4} - \frac{e^{-.0}}{-.4}$$

   Finally, we take the limit. **What is the limit here?**

   (d) The integral converges if the limit exists, and diverges otherwise. **Does this converge?**

2. Next examine $\int_0^1 \frac{1}{\sqrt{x}}\,dx$
   (a) **Sketch a rough plot**. Then **shade in the area** that represents the integral from 0 to 1.

   (b) This integral is improper because the function has an infinite discontinuity ($f(x)$ has a vertical asymptote on the region of integration). **What value causes this integral to be improper**?
   $x =$ ____________

   (c) Set up the integral using a limit for the infinite discontinuity.

   (d) Integrate

   (e) Plug in the endpoints

   (f) Take the limit. What is the limit here?

   (g) The integral converges if the limit exists, and diverges otherwise. Does it converge?

[1] *Active Calculus* by Matt Boelkins, David Austin and Steven Schlicker