9.4 Series
Group Work Target Practice Solutions

1. Go through each and every one of our series tests in 9.2–9.4, as well as the conditions for the tests, and write down whether each can be applied or not, and if so, whether the test determines convergence. Show work/reasoning: $\sum_{n=1}^{\infty} \frac{n^2}{n^3+6}$

   (a) Is this a geometric series?

   no

   not a constant ratio from one term to the next

   (b) Can we apply the Terms not Getting Smaller?

   no

   the terms do go to 0, so we must use another test

   (c) Are the terms decreasing and positive eventually, and if so is this an integral we can do?

   yes.

   The terms are positive, and eventually decreasing, so we can evaluate the integral by using w-subs, with $w = n^3 + 6$, $dw = 3n^2$, so $\frac{dw}{3} = n^2$:

   $\lim_{b->\infty} \int^b \frac{dw}{3w} = \lim_{b->\infty} \frac{1}{3} ln|w| \Big|^b = \lim_{b->\infty} \frac{1}{3} ln|n^3+6| \Big|^b$. The integral diverges, so the series does too.

   (d) Are the terms positive, and if so, can we directly compare the series with another (smaller than a convergent series, larger than a divergent series)?

   no

   The terms are positive but the natural direct comparison would be to $\frac{n^2}{n^3+6} < \frac{n^2}{n^3} = \frac{1}{n}$. However, the harmonic series $\sum \frac{1}{n}$ is divergent by the integral test, so it doesn't help us to be smaller than a divergent series. So the test is not useful for direct comparison (however, limit comparison will work, as below).

   (e) Are the terms positive, and if so, can we limit compare the series with another to obtain $0 < \lim_{n->\infty} \frac{a_n}{b_n} < \infty$?

   yes

   The terms are positive, and we can limit compare with $\sum \frac{1}{n}$:

   $\lim_{n\to\infty} \frac{\frac{n^2}{n^3+6}}{\frac{1}{n}} = \lim_{n\to\infty} \frac{n^3}{n^3+6} = \lim_{n\to\infty} \frac{1}{1+\frac{6}{n^3}} = 1$. Since the limit is 1, it falls within $0 < \lim_{n->\infty} \frac{a_n}{b_n} < \infty$, so the series behaves the same way as $\sum \frac{1}{n}$, which diverges by the integral test: (terms are positive and decreasing, integral gives $\lim_{b\to\infty} ln|b| - ln|1|$, which diverges).

(f) Does the ratio test give us $L < 1$ or $L > 1$?

no. The ratio test fails because $L = 1$:

$$\frac{\frac{(n+1)^2}{(n+1)^3+6}}{\frac{n^2}{n^3+6}} = \frac{n^2+2n+1}{n^3+3n^2+3n+1+6}\frac{n^3+6}{n^2} = \frac{n^5+...}{n^5+...}$$

Divide by the highest power n^5. Then the limit will be 1 and hence the ratio test does not tell us anything, and we must use another test.

(g) Is this an alternating series?

no the terms do not alternate in sign from positive to negative

In Maple we can see that the series diverges even though the terms go to 0, and the terms are decreasing after 3.

Given the following sequences and series, determine if they converge or diverge and EXPLAIN or SHOW WORK documenting why your answer is correct. If more than one test applies, choose whichever you prefer. List the test and why it works.

2. $\displaystyle\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n+5}$

This series diverges by the Terms not Getting Smaller test.

$\lim_{n\to\infty}(-1)^{n-1}\frac{n}{n+5} = \pm 1 \neq 0$ so the series diverges

3. $\displaystyle\sum_{n=0}^{\infty}\frac{n}{n!}$

This series converges by the ratio test.

$$\frac{\frac{(n+1)}{(n+1)!}}{\frac{n}{n!}} = \frac{n!}{(n+1)!}\frac{n+1}{n} = \frac{1}{(n+1)}\frac{n+1}{n} = \frac{1}{n}$$

So $L = \lim_{n\to\infty}\frac{a_{n+1}}{a_n} = \lim_{n\to\infty}\frac{1}{n} = 0$ and hence the ratio test tells us the series converges (because $L < 1$).

4. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1+n}{2n}\right)^n$. Hint: $\lim_{n\to\infty}\left(\frac{1+n}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

The hint indicates that we should use limit comparison test with $\sum\left(\frac{1}{2}\right)^n$, which converges because it is a geometric series with $|x| = \frac{1}{2} < 1$

The terms of $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1+n}{2n}\right)^n$ are positive, so examine

$$\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{\left(\frac{1+n}{2n}\right)^n}{\left(\frac{1}{2}\right)^n} = \lim_{n\to\infty} \frac{\left(\frac{1}{2}\right)^n \left(\frac{1+n}{n}\right)^n}{\left(\frac{1}{2}\right)^n} = \lim_{n\to\infty} \left(\frac{1+n}{n}\right)^n = \lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$$

Since the limit is positive, not 0 or ∞, the series behaves the same way $\sum \left(\frac{1}{2}\right)^n$, so it converges by limit comparison test.

5. $a_i = \dfrac{1}{i}$

   This is a sequence (not a series) and the terms converge to 0. As i gets bigger $\frac{1}{i}$ gets smaller and smaller and approaches 0.

6. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$

   The terms are positive.

   We'll use direct comparison test with the divergent harmonic series $\sum \frac{1}{n}$ (which diverges by the integral test, as the integral turns into $\displaystyle\lim_{n\to\infty} \int_2^b \frac{dn}{n} = \lim_{n\to\infty} \ln|b| - \ln|2|$.)

   Notice that $\ln(n) \leq n$, so $\frac{1}{\ln(n)} \geq \frac{1}{n}$

   So $\displaystyle\sum_{n=2}^{\infty} \frac{1}{\ln(n)}$ diverges by comparison, since it is larger than a diverging series.

7. Find an estimate of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ to within 0.01, if it converges.

   Yes the series converges by the alternating series test, because the terms are alternating and decreasing. Or we could have used the absolute value test, because $\sum \frac{1}{n^2}$ converges by the integral test. The truncation error of using S_n, the partial sum, to approximate an alternate series, is less than the absolute value of next term in the series a_{n+1}.

   We want n so that $|S - S_n| \leq .01 = \frac{1}{100}$.

   Notice $|S - S_n| \leq a_{n+1} = \frac{1}{(n+1)^2}$, so we want $\frac{1}{(n+1)^2} \leq \frac{1}{100} = \frac{1}{10^2}$.

   Solve for n: $\frac{1}{(n+1)^2} \leq \frac{1}{10^2}$

   $n+1 \geq 10$, so $n \geq 9$ and $\sum_{n=1}^{9} \frac{(-1)^{n-1}}{n^2} = .82796$