9.4 Series
Group Work Target Practice Solutions

1. Go through each and every one of our series tests in 9.2–9.4, as well as the conditions for the tests, and write down whether each can be applied or not, and if so, whether the test determines convergence. Show work/reasoning: $\sum_{n=1}^{\infty} \frac{n^2}{n^3+6}$

   (a) Is this a geometric series?

   no

   not a constant ratio from one term to the next like $\sum 2^n$ would have

   (b) Can we apply the Terms not Going to 0 $\sum a_n$, $a_n \not\to 0$?

   no

   the terms do go to 0, so we must use another test [can use L'Hôpital's to see, for example]

   (c) Are the terms decreasing and positive eventually, and if so is this an integral we can do?

   yes.

   The terms are positive, and eventually decreasing. You can see they are decreasing eventually numerically, graphically, or by taking the derivative and showing it is negative after a few terms of the series. The derivative is obtained by quotient rule:

   $$\frac{(n^3+6)2n - n^2(3n^2)}{n^3+6^2} = \frac{2n^4+12n-3n^4}{n^3+6^2} = \frac{-n(n^2-12)}{n^3+6^2}$$

   The derivative is positive for n=1 and 2, but negative from n=3 onward, so the series is eventually decreasing (starting at $n = 3$).

   Thus we can evaluate the integral by using w-subs, with $w = n^3 + 6$, $dw = 3n^2$, so $\frac{dw}{3} = n^2$:

   $\lim_{b->\infty} \int^b \frac{dw}{3w} = \lim_{b->\infty} \frac{1}{3} ln|w| \Big|^b = \lim_{b->\infty} \frac{1}{3} ln|n^3+6| \Big|^b$. The integral diverges, so the series does too.

   (d) Are the terms positive, and if so, can we limit compare the series with another to obtain $0 < \lim_{n->\infty} \frac{a_n}{b_n} < \infty$?

   yes

   First note that the terms are positive and the natural direct comparison would be to $\frac{n^2}{n^3+6} < \frac{n^2}{n^3} = \frac{1}{n}$. However, the harmonic series $\sum \frac{1}{n}$ is divergent by the integral test, so it doesn't help us to be smaller than a divergent series. So the test is not useful for direct comparison (however, limit comparison, the more powerful test we are focusing on, will work, as below).

   The terms are positive, and we can limit compare with $\sum \frac{1}{n}$:

$\lim\limits_{n\to\infty} \frac{\frac{n^2}{n^3+6}}{\frac{1}{n}} = \lim\limits_{n\to\infty} \frac{n^3}{n^3+6} = \lim\limits_{n\to\infty} \frac{1}{1+\frac{6}{n^3}} = 1$. Since the limit is 1, it falls within $0 < \lim\limits_{n->\infty} \frac{a_n}{b_n} < \infty$, so the series behaves the same way as $\sum \frac{1}{n}$, which diverges by the integral test: (terms are positive and decreasing, integral gives $\lim_{b\to\infty} ln|b| - ln|1|$, which diverges).

(e) Does the ratio test give us $L < 1$ or $L > 1$?

no. We use ratio test when we see factorials or exponentials like 2^n that aren't already geometric, not for powers of n like n^3. The ratio test fails because, after a lot of algebra $L = 1$:

$L = \lim_{n\to\infty} \frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty} \frac{\frac{(n+1)^2}{(n+1)^3+6}}{\frac{n^2}{n^3+6}} = \lim\limits_{n\to\infty} \frac{\frac{n^2+2n+1}{n^3+3n^2+3n+1+6}}{\frac{n^2}{n^3+6}} = \lim_{n\to\infty} \frac{n^2+2n+1}{n^3+3n^2+3n+1+6} \frac{n^3}{n}$

$= \lim_{n\to\infty} \frac{n^5+...}{n^5+...}$

Then repeated L'Hôpital's or similar will show that the limit will be 1 and hence the ratio test does not tell us anything, and we must use another test.

(f) Is this an alternating series?

no the terms do not alternate in sign from positive to negative

In Maple we can see that the series diverges even though the terms go to 0.

For sequences EXPLAIN or SHOW WORK documenting why your answer is correct:

(a) does it converge or diverge, and why

(b) what is the limit if it converges?

(c) show work for L'Hôpital's Rule if it applies.

For series EXPLAIN or SHOW WORK documenting why your answer is correct:

(a) (LG 3) choose a series test we can successfully use on it from among geometric series, terms not going to 0, linearity, or integral test

(b) fully document why the series test works, including any assumptions

(c) specify whether the series converges or diverges, and why

2. $\sum_{n=1}^{\infty}(-1)^{n-1}\frac{n}{n+5}$

This series diverges by the Terms not Going to 0 $\sum a_n$, $a_n \not\to 0$ test. Look at $\lim_{n\to\infty}(-1)^{n-1}\frac{n}{n+5}$. You can use

divide by highest power to obtain $(-1)^{n-1}\frac{1}{1+\frac{5}{n}}$. This limit does not exist because it alternates towards ± 1 as n is large, and "does not exist" is not 0 , so $a_n \not\to 0$ and the series diverges.

What doesn't work and why?

Note that the alternating series test does not apply because even though it is an alternating series, the absolute value of the terms is not decreasing. Similarly the integral test does not apply because the terms are not decreasing. The terms aren't positive, so that rules out limit comparison (and also integral test).

This is not a geometric series since there is no fixed ratio between successive terms like $\sum 2^n$ would have. And linearity doesn't apply either as it is not the sum of two known series.

Since the terms $\not\to 0$ already applies we wouldn't go to the ratio test. Plus ratio test works the best for factorials or exponentials like 2^n that aren't already geometric, so if it is not this form and another test works, don't use ratio test on other functions, since it is likely to fail. If you did try ratio test here that would be wasted algebra. The absolute values of the successive terms means the powers of -1 go away and we would have:

$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{\frac{(n+1)}{(n+1)+5}}{\frac{n}{n+5}} = \lim_{n\to\infty}\frac{n+1}{n+6}\ \frac{n+5}{n} = \lim\limits_{n\to\infty}\frac{n^2+6n+5}{n^2+6n} = \lim\limits_{n\to\infty}\frac{2n+6}{2n+6} = 1$ so the ratio test is inconclusive and the test fails.

3. $\sum_{n=0}^{\infty}\frac{n}{n!}$

This series converges by the ratio test. The presence of the factorial leads naturally to the ratio test as ratio test works the best for factorials or exponentials like 2^n that aren't already geometric.

$L = \lim_{n\to\infty}\frac{|a_{n+1}|}{|a_n|} = \lim_{n\to\infty}\frac{\frac{(n+1)}{(n+1)!}}{\frac{n}{n!}} = \lim\limits_{n\to\infty}\frac{\frac{(n+1)}{n}}{\frac{(n+1)!}{n!}} = \lim\limits_{n\to\infty}\frac{\frac{(n+1)}{n}}{\frac{(n+1)n!}{n!}} = \lim_{n\to\infty}\frac{\frac{(n+1)}{n}}{(n+1)} = \lim\limits_{n\to\infty}\frac{\frac{n+1}{n+1}}{n} = \lim\limits_{n\to\infty}\frac{1}{n} = 0$

So $L = 0$ and hence the ratio test tells us the series converges because $L < 1$.

What doesn't work and why?

This is not geometric as there is no fixed ratio between successive terms like $\sum 2^n$ would have.

The terms do go to 0 as $\lim_{n\to\infty} \frac{n}{n!} = \lim_{n\to\infty} \frac{n}{n(n-1)!} = \lim_{n\to\infty} \frac{1}{(n-1)!} = 0$ so the terms $\not\to 0$ test is inconclusive and we must use another test.

There is no sum of two series to apply linearity to.

The integral test won't work on factorials as we can't integrate factorials (Side note: there is a generalization of factorial that is beyond the scope of the class related to a gamma function Γ which one can integrate and is used in computations such as statistics, for those that are interested)

Our use of limit comparison has mostly been to quotients of polynomials and this isn't such a quotient.

It is not an alternating series from positive to negative term by term nor negative to positive term by term.

4. $a_i = \dfrac{1}{i}$

This is a sequence (not a series) and the terms converge to 0. As i gets bigger $\frac{1}{i}$ gets smaller and smaller and approaches 0. You can argue numerically like this or draw a sketch of the function which is concave up and decreasing for positive values.

What doesn't work and why?

Since it isn't a series, none of the series tests apply.

5. Find an estimate of $\displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n^2}$ to within 0.01, if it converges.

Yes the series converges by the alternating series test, because it is an alternating series where the terms are alternating and decreasing. The truncation error of using S_n, the partial sum, to approximate an alternate series, is less than the absolute value of next term in the series, i.e. a_{n+1}.

We want n so that $|S - S_n| \leq .01 = \frac{1}{100}$.

Notice $|S - S_n| \leq a_{n+1} = \frac{1}{(n+1)^2}$, so we want $\frac{1}{(n+1)^2} \leq \frac{1}{100} = \frac{1}{10^2}$.

Solve for n: $\frac{1}{(n+1)^2} \leq \frac{1}{10^2}$

$n + 1 \geq 10$, so $n \geq 9$ and $\sum_{n=1}^{9} \frac{(-1)^{n-1}}{n^2} = .82796$

This is very useful in mathematics and computer science contexts.

6. $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1+n}{2n}\right)^n$. Hint: $\lim_{n\to\infty}\left(\frac{1+n}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$.

The hint indicates that we should use limit comparison test with $\sum\left(\frac{1}{2}\right)^n$, which converges because it is a geometric series with $|x| = \frac{1}{2} < 1$

The terms of $\displaystyle\sum_{n=1}^{\infty}\left(\frac{1+n}{2n}\right)^n$ are positive, so examine

$$\lim_{n\to\infty}\frac{a_n}{b_n} = \lim_{n\to\infty}\frac{\left(\frac{1+n}{2n}\right)^n}{\left(\frac{1}{2}\right)^n} = \lim_{n\to\infty}\frac{\left(\frac{1}{2}\right)^n\left(\frac{1+n}{n}\right)^n}{\left(\frac{1}{2}\right)^n} = \lim_{n\to\infty}\left(\frac{1+n}{n}\right)^n = \lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n = e$$

Since the limit is positive, not 0 or ∞, the series behaves the same way $\sum\left(\frac{1}{2}\right)^n$, so it converges by limit comparison test.