

Differential Equations and Slope Fields in Maple

Goal: Explore differential equations using a variety of representations (numerical, symbolic, and graphical) through pattern exploration assisted by appropriate technology, including the computer algebra system Maple, which is one of the course goals.

Read the text and hit return in each Maple command line (the commands are in red).

In 11.1 we check whether a given function is a solution of a differential equation, an equation involving derivatives or differentials, (or whether initial conditions work) by taking derivatives and testing by plugging in to see if we satisfy the differential equation.

In 11.2 we use visualizations of slopes in order to create our own solutions. A slope field is a set of signposts directing us across the plane. For example, a slope of 1 at a point has a tangent line that is 45 degrees from the horizontal and is directed upward to the right because of the positive slope. We can plug the coordinates of a point (x,y) or (t,y) into a differential equation to obtain a numerical value and sketch a direction from there.

1. By-hand checking a solution (11.1)

Check that $y = e^{t+3}$ is a solution of the differential equation $\frac{dy}{dt} = y$ --- plug in e^{t+3} every place you

see a y in the equation. $\frac{d(e^{t+3})}{dt} = e^{t+3}$. Take the derivative of the left side (don't forget to use chain rule!). Show work in your notes.

The derivative of e^{t+3} with respect to t is $e^{t+3} \cdot (1 + 0) = e^{t+3}$ by chain rule, so the left and right sides of the equation are indeed equal. Thus $y = e^{t+3}$ is a solution.

In fact, any constant $e^{t+constant}$ is also a solution of this differential equation.

2. By-hand checking an initial condition (11.1)

Can $y = e^{t+3}$ pass through the point $(t,y) = (0, -1)$?

We plug in for t and y to see whether $-1 = e^{0+3}$ is possible. Is it possible? Write your response in your notes.

No because -1 is not e^3 so that specific solution can not pass through that point. However, we could change the 3 to obtain a solution that would. d

3. Graphically generating a slope by-hand (11.2)

While the above solution can't pass through the point $(0,-1)$, there are other solutions to this DE that can pass through $(0,-1)$. We'll graphically generate the start of what that solution would look like very very close to that point by creating a graphical tangent line.

First plug in to the right hand side of the differential equation (y in this case, since

$$\frac{dy}{dt} = y \leftarrow \text{plug in}$$

here) to compute the numerical slope at the point $(t,y) = (0,-1)$. Notice that -1 is what we need to plug in.

Then sketch by-hand the tangent line corresponding to $\frac{dy}{dt} = -1$.

The slope at $(0,-1)$ is given as y by the differential equation, which is -1 here, so the tangent line is -45° negative slope down to right.

Note that as we talked about with Taylor series, if we are very very close to the center, a Taylor polynomial (like the linear approximation) looks like the function because of the Lagrange Error estimate/Taylor's Theorem. We investigate more on this issue in 11.3 via numerical approximations in Euler's method, but it applies here since we only sketch a small tangent line.

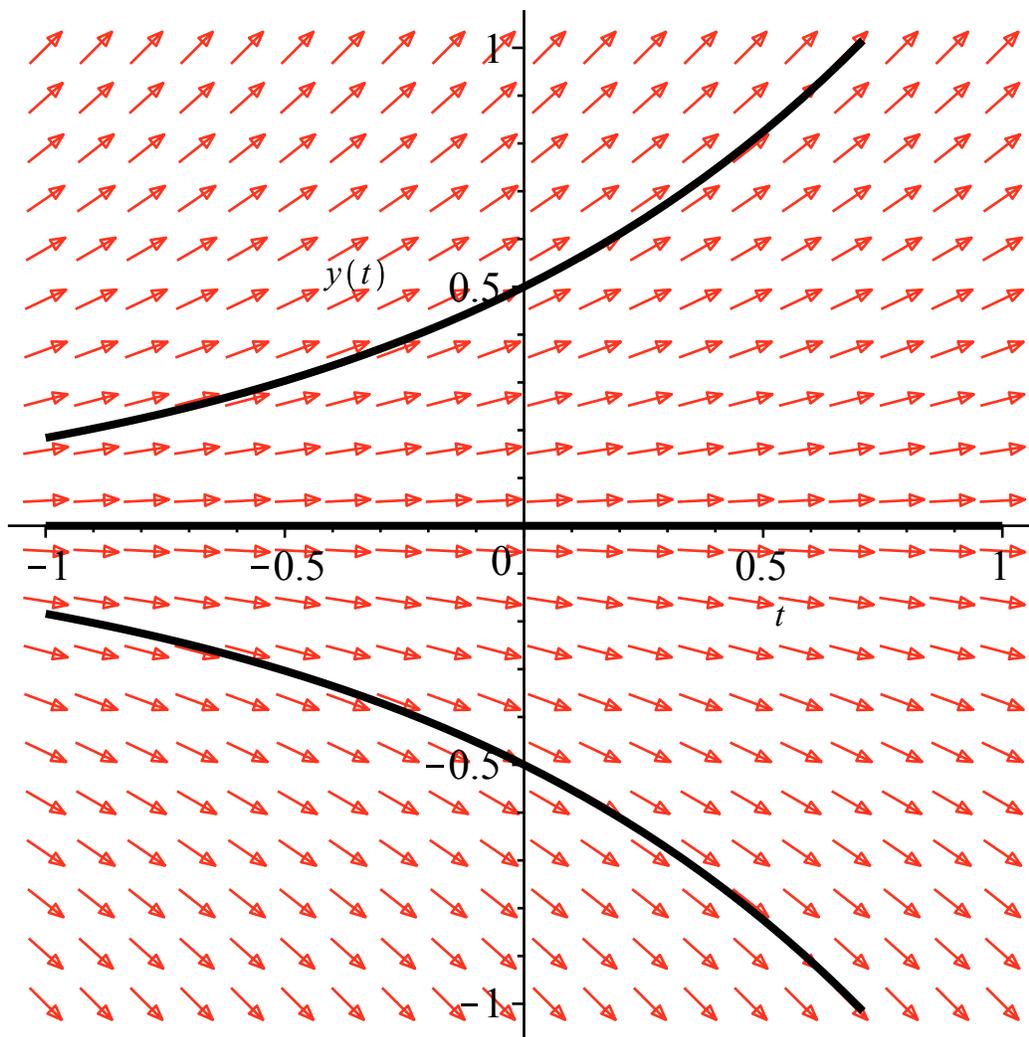
The remaining worksheet combines aspects from 11.1 and 11.2:

4. Maple's slope diagram and solution for $\frac{dy}{dt} = y$

Maple can create slope field diagrams very quickly using the DEplot package, which relies on a variable t in place of x that the book uses.

Notice that in the code below, $\text{diff}(y(t),t)=y(t)$ is used to represent the differential equation $\frac{dy}{dt} = y$.

```
> with(DEtools):  
DEplot(diff(y(t),t) = y(t), y, t = -1 .. 1, y = -1 .. 1, [y(0) =  
.5, y(0) = -.5, y(0) = 0], arrows = medium, linecolor = black);
```



Compare the what you drew in #3 with what Maple drew near $y = -1$. They should look similar.

Yes they do! Look at the point $(0, -1)$ at the bottom middle of the graph to see the tangent line is -45° negative slope down to right.

Each slope vector represents the direction and magnitude:

horizontal line has slope 0.

vertical line has infinite slope.

45° positive slope up to right has a slope of 1.

-45° negative slope down to right has a slope of -1 , like in #3.

Other slopes fall somewhere in between these, i.e. a slope of 2 is a bit steeper than the 45° angle and still slopes up the right right.

The solution to the differential equation is called an **equilibrium** solution if the derivative is zero everywhere, represented as a horizontal line. We see that $y=0$ is an equilibrium solution here. It is classified as **unstable**, because a small perturbation off of it results in solutions that move away from it in all directions as we move to the right in time. On the other hand, **stable** equilibrium solutions would have solutions nearby tending towards them as we move to the right, as we'll see below. Sometimes equilibrium solutions are **neither** stable nor unstable.

Maple can also solve differential equations for algebraic solutions. Execute the following:

```
> dsolve(diff(y(t),t) = y(t));
```

$$y(t) = _C1 e^t \quad (1)$$

Maple gives us $_C1$, a generic constant. While this looks a bit different than e^{t+3} from 1. The solutions are algebraically equivalent because of algebra: $e^{t+constant} = e^t e^{constant} = e^{constant} e^t$ (see the powers of x of the Algebra, Geometry & Trigonometry portion of the review sheet). Now $e^{constant}$ is also a constant. So in this case Maple's $_C1$ represents our $e^{constant}$ giving $_C1 e^t$.

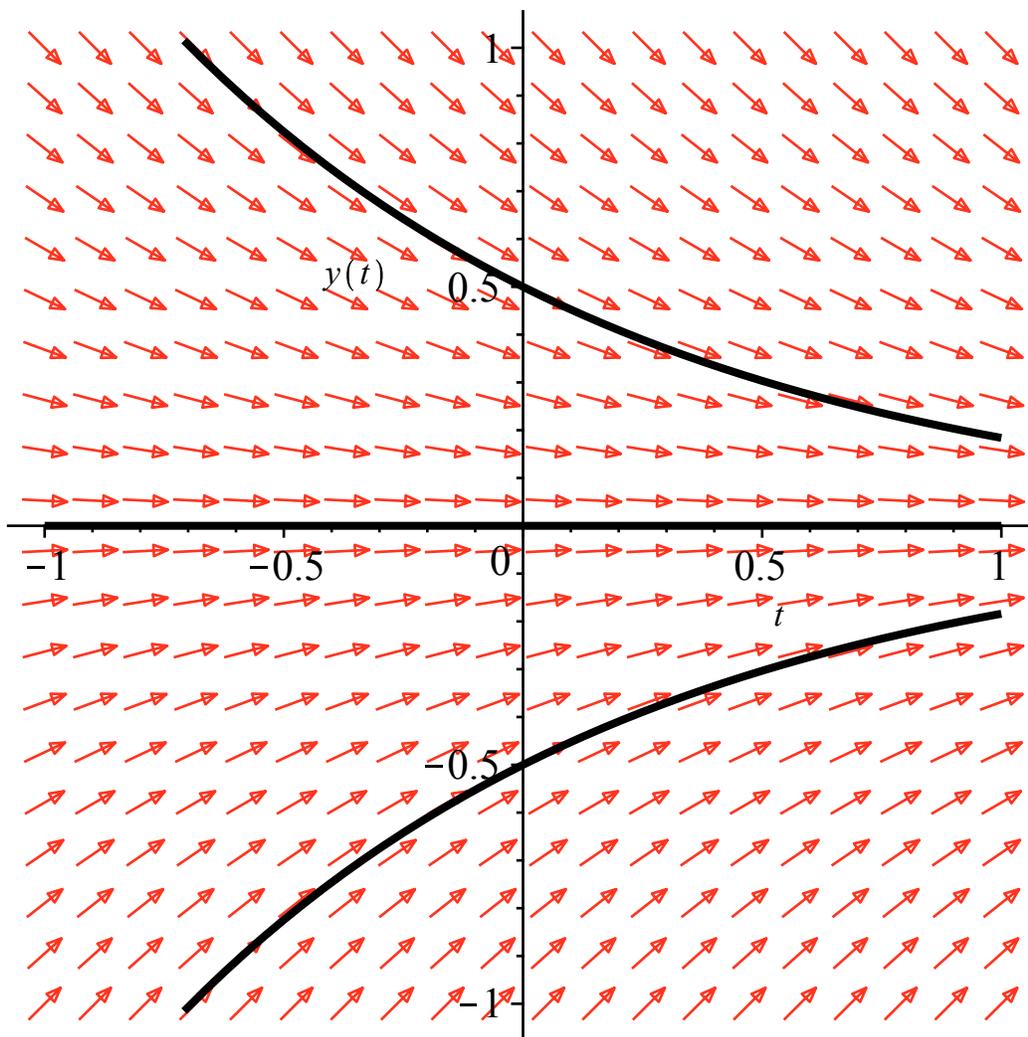
Ask me any questions you have on the terminology or algebraic equivalences.

5. Interpreting Maple's slope diagram for Equilibrium Solutions of

$$\frac{dy}{dt} = -y$$

In the command below I modified the differential equation to add one negative to the equal portion of the DE ($= -y(t)$ replaces $=y(t)$). The other instance of $y(t)$ in the command remains the same. Execute:

```
> DEplot(diff(y(t),t) = -y(t), y, t = -1 .. 1, y = -1 .. 1, [y(0) =  
.5, y(0) = -.5, y(0) = 0], arrows = medium, linecolor = black);
```



Notice that once we added the negative sign in the above command the DEplot changed.

Are there any equilibrium solutions?

If so, are they stable, unstable, or neither?

$y=0$ is still an equilibrium solution since it results in a constant horizontal solution $y'=0$. We can see it is a stable solution, because other solutions nearby are heading towards it.

6. By-hand checking a solution of $\frac{dy}{dt} = -y$

Is $C e^{-t}$ a solution of this differential equation, where C is a constant? In your notes, plug in $C e^{-t}$ every place you see a y in $\frac{dy}{dt} = -y$ and then take the derivative to see if the equation works.

We check $\frac{d(C e^{-t})}{dt} = -C e^{-t}$ to see if the equation holds. Taking the derivative using chain rule we see that it is $C e^{-t}(-1) = -C e^{-t}$ so the differential equation is satisfied and $C e^{-t}$ is a solution.

7. Fixing a Problem with a Singularity in

$$\frac{dy}{dt} = \frac{y}{t} \text{ at } t=0.$$

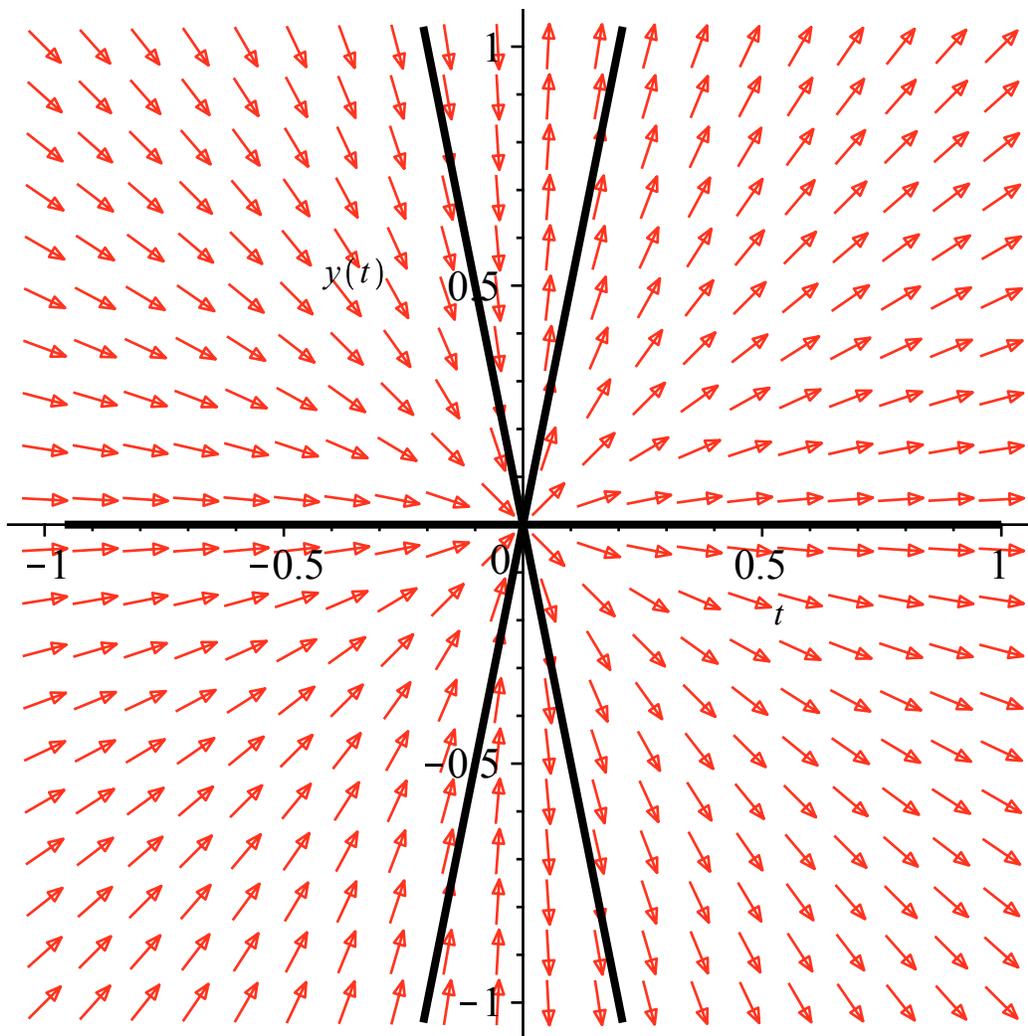
Execute the differential equation in the command below. Notice that I replaced $y(t)$ with $y(t)/t$.

```
> DEplot(diff(y(t),t) = y(t)/t, y, t = -1 .. 1, y = -1 .. 1, [y(0) =
.5, y(0) = -.5, y(0) = 0], arrows = medium, linecolor = black);
Error, (in dsolve/numeric/make_proc) ode system is singular at the
initial point
```

Activity: We receive an error message because we have undefined slopes when t (or x) is 0, and the initial conditions all specify $y(0)$.

Create a small perturbation of the problematic initial conditions by replacing all three instances of the 0 within $y(0)$ inside the square brackets with $y(.1)$ in order to remove the error message and obtain the DEplot.

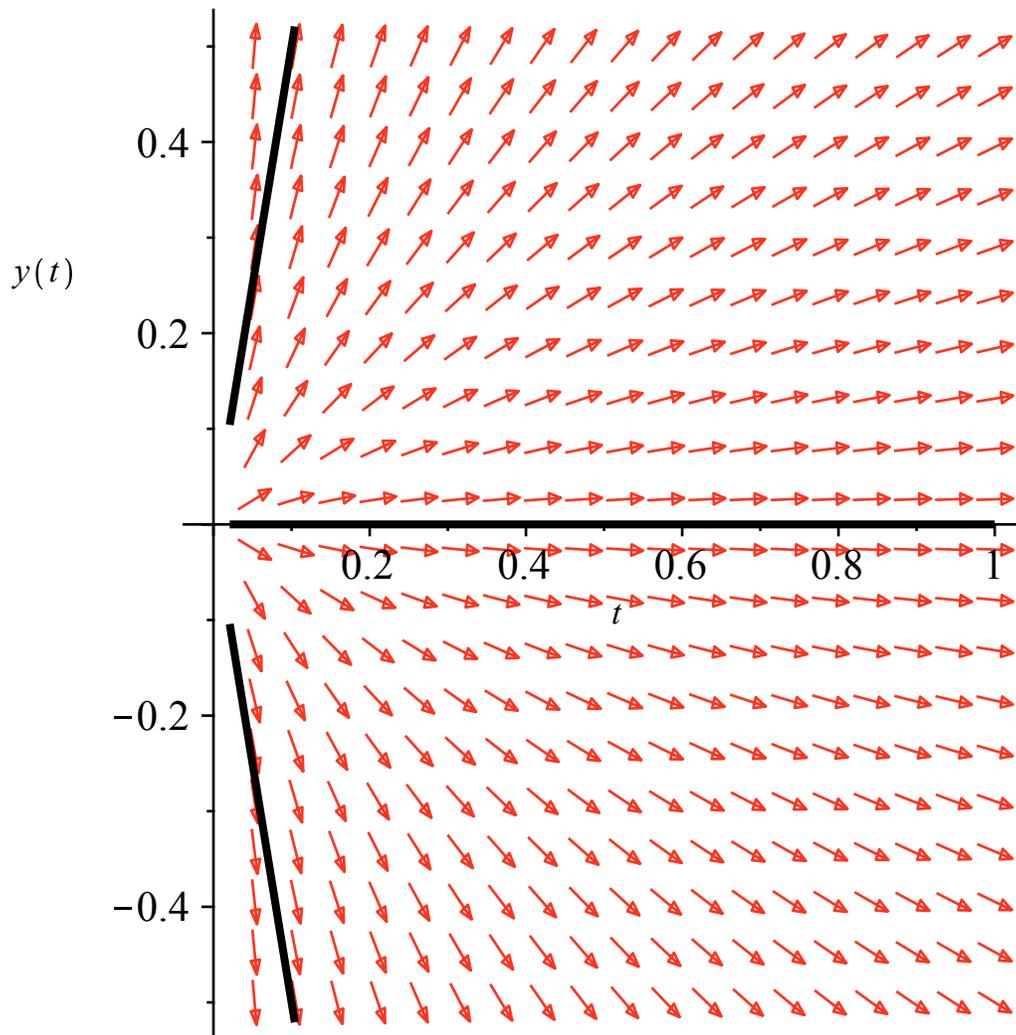
```
> DEplot(diff(y(t),t) = y(t)/t, y, t = -1 .. 1, y = -1 .. 1, [y(.1) =
.5, y(.1) = -.5, y(.1) = 0], arrows = medium, linecolor = black);
```

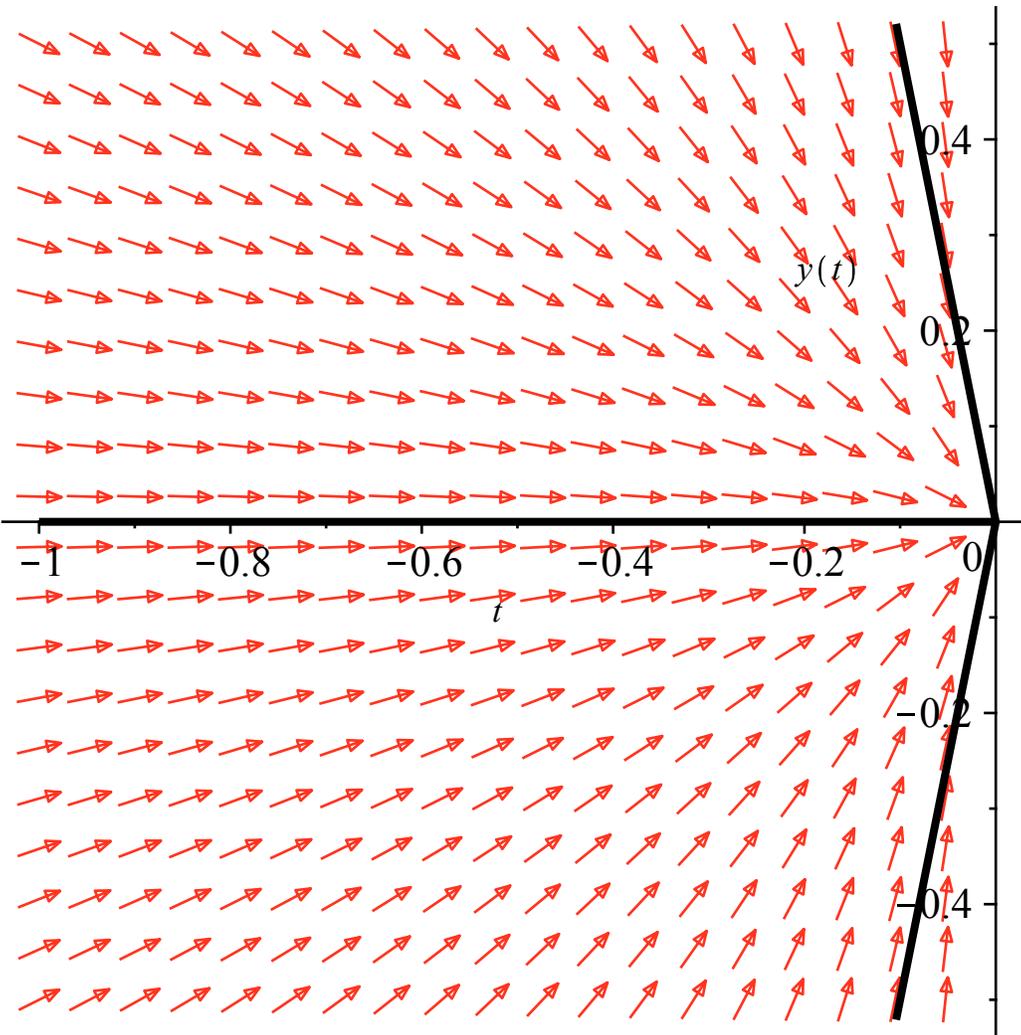


Notice that there is an equilibrium solution for $\frac{dy}{dx} = \frac{y}{t}$ when $y=0$ as we see the constant horizontal line there ($t=0$ was the problem above, not $y=0$ which is defined everywhere except when $t=0$). To the right

of the y -axis, solutions that start nearby $y=0$ move away from it, but they do the opposite to the left of the y -axis as they move towards the horizontal line. So this **neither** a stable nor unstable solution.

Solutions move away from the equilibrium in the first picture as we move to the right, and towards it in the second as we move to the right:

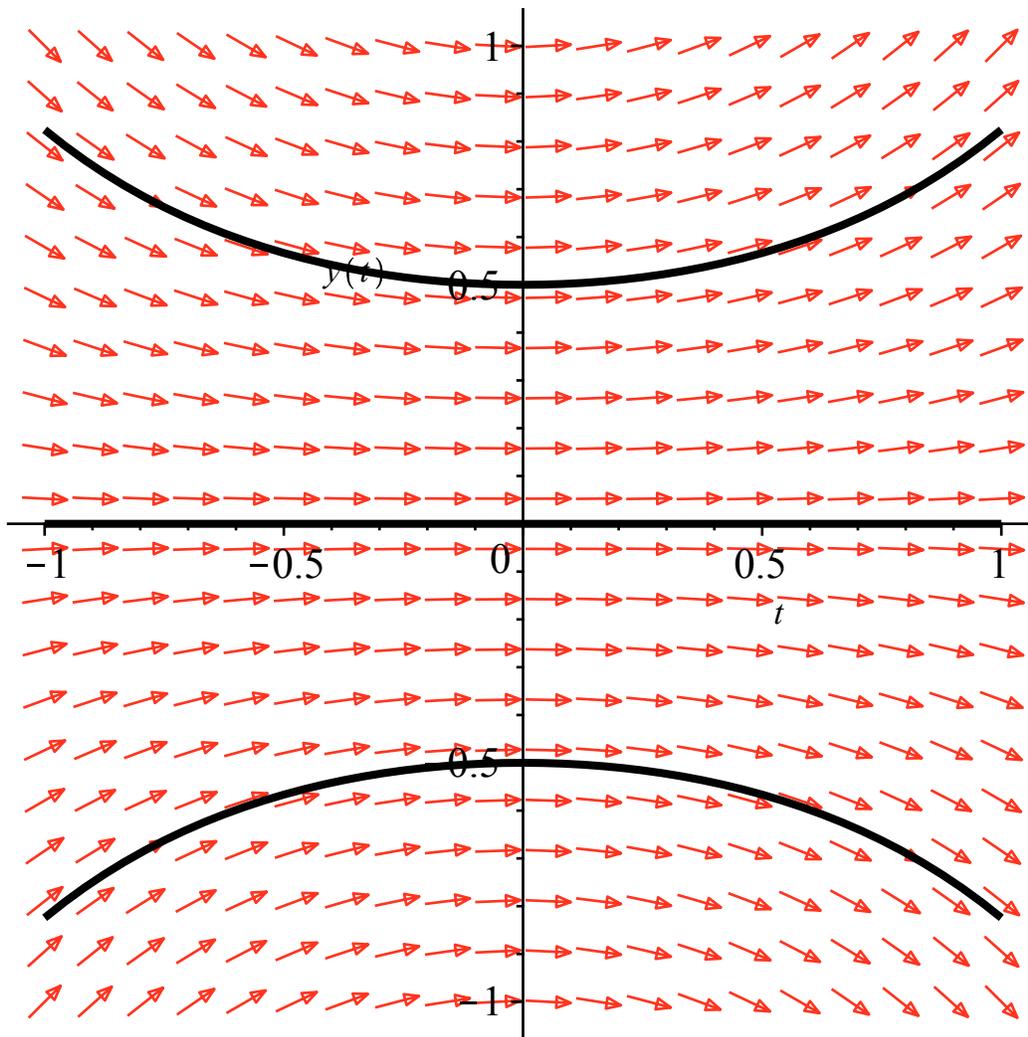




8. Modifying Maple's DEplot command for $\frac{dy}{dt} = ty$.

Activity: Modify the differential equation in the command below. **Replace** $y(t)$ with $-t*y(t)$

```
> DEplot(diff(y(t),t) = t*y(t), y, t = -1 .. 1, y = -1 .. 1, [y(0) =
.5, y(0) = -.5, y(0) = 0], arrows = medium, linecolor = black);
```



9. Interpreting Maple's slope diagram for Equilibrium Solutions of

$$\frac{dy}{dt} = ty.$$

Are there any equilibrium solutions? Is it stable (nearby solutions always tend towards it as we move to the right), unstable (nearby solutions always tend away from it as we move to the right), or neither (a combination). Hint: check to the right of the y -axis for the behaviour of those nearby solutions, and then to the left of the y -axis, and see if they match.

$y=0$ is an equilibrium solution, but it is neither stable nor unstable. To the left of the y -axis solutions tend towards $y=0$, but to the right of the y -axis they go away from it.

10. $\frac{dy}{dt} = \text{your creation!}$

We can use all sorts of functions although they must be expressed like Maple does, like

$$=(y(t))^2$$

$$=\sec(y(t))$$

$$=t*\sec(y(t))$$

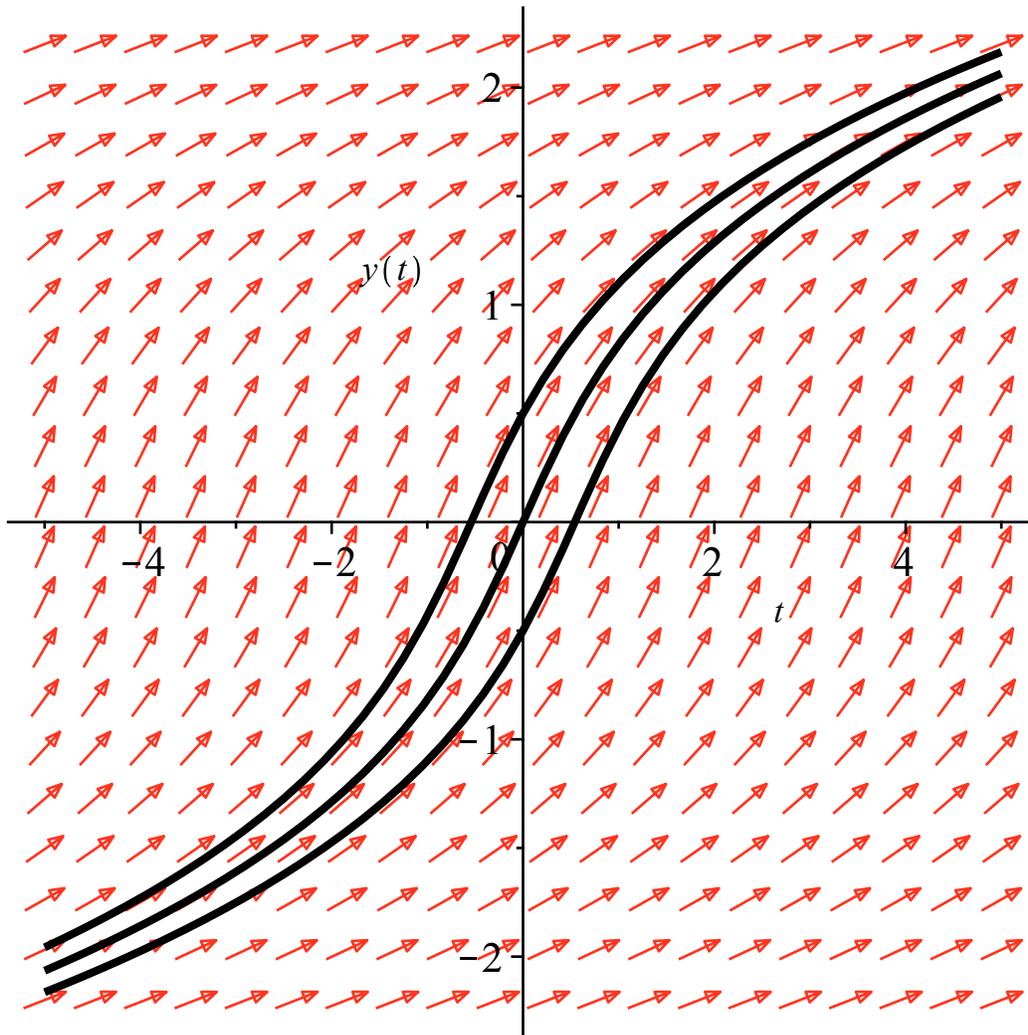
$$=\exp(t)*\sec(y(t))$$

for instance

Activity: Modify the differential equation in the command below. Replace $y(t)$ with a differential equation of your creation using t and/or $y(t)$. It could be one of the ones that I just listed, or something else completely.

Here I am using $\frac{dy}{dt} = \frac{1}{1+y^2}$ with the idea that if we wanted to integrate the right side, that would look something like $\arctan(y)$, which it does:

```
> DEplot(diff(y(t),t) = 1/(1+ y(t)*y(t)), y, t = -5 .. 5, y = -2.2 .. 2.2, [y(0) = .5, y(0) = -.5, y(0) = 0], arrows = medium, linecolor = black);
```



If Maple gives you an error then you might need to modify the initial conditions or domain or range if your function has undefined values.

List the DE in your notes.

11. If there is time, go back to the class highlights webpage for the CoursEval link. If you don't have time today, there will be time in an upcoming lab.