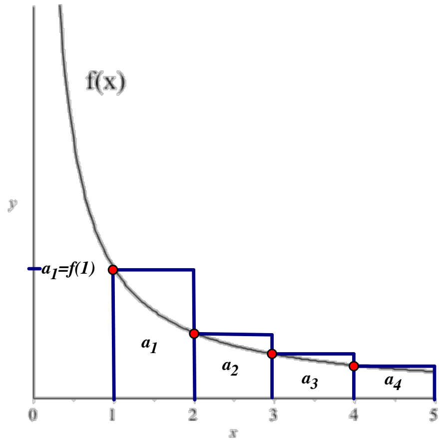
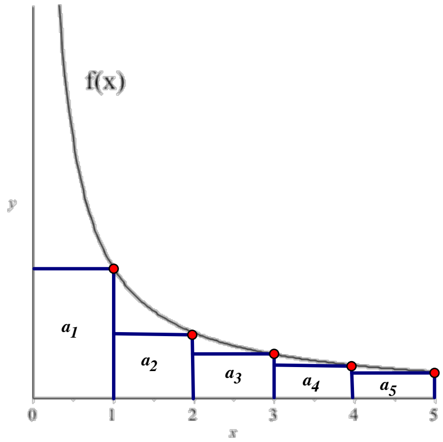
9.3 Integral Test

Goal: How do we determine if a series that isn't a geometric series converges?

Suppose $a_n = f(n)$, where $f(x)$ is decreasing and positive. We will consider $\sum_{n=1}^{\infty} a_n$.

1. **Draw the Left Sum** with $\Delta x = 1$ starting at $x = 1$ — **Draw the Right Sum** with $\Delta x = 1$ starting at $x = 0$





2. In each box, put the element of the sequence a_n that gives the area of that box. Use the fact that $a_n = f(n)$ and $\Delta x = 1$.

   Notice that the width of each rectangle is $\Delta x = 1$, so the area is the same as the height.

3. What relationship between the improper integral $\int_1^{\infty} f(x)\, dx$ and the sum $\sum_{n=1}^{\infty} a_n$ is illustrated by the picture on the left?

   $$\int_1^{\infty} f(x)\, dx \leq \sum_{n=1}^{\infty} a_n$$

   by the picture on the right?

   $$\int_1^{\infty} f(x)\, dx \geq \sum_{n=2}^{\infty} a_n \text{ so } \int_1^{\infty} f(x)\, dx + a_1 \geq \sum_{n=1}^{\infty} a_n$$

4. Evaluate $\int_1^{\infty} \frac{1}{x}\, dx$.

   $$\int_1^{\infty} \frac{1}{x}\, dx = \lim_{b\to\infty} \int_1^{b} \frac{1}{x}\, dx = \lim_{b\to\infty} \ln x \Big|_1^b = \lim_{b\to\infty} \ln b - \ln 1 = \lim_{b\to\infty} \ln b, \text{ which diverges.}$$

5. Determine if $\sum_{n=1}^{\infty} \frac{1}{n}$ converges or diverges.

   $\int_1^{\infty} f(x)\, dx \leq \sum_{n=1}^{\infty} a_n$. The terms of the series are decreasing and and positive, so the series behaves the same as the integral, which diverges.

Example: Use the integral test to show that $\sum_{n=1}^{\infty} \frac{1}{n^5}$ converges and give bounds on the sum.

$$\int_1^{\infty} \frac{1}{x^5}\, dx = \lim_{b\to\infty} \int_1^b x^{-5}\, dx = \lim_{b\to\infty} \frac{x^{-5+1}}{-5+1}\Big|_1^b = \frac{1}{-4}\lim_{b\to\infty} \frac{1}{x^4}\Big|_1^b = \frac{1}{-4}\lim_{b\to\infty}(\frac{1}{b^{-4}} - \frac{1}{1^4}) = \frac{1}{-4}(0-1) = \frac{1}{4}$$

We also have bounds on the sum:

$\frac{1}{4} \le \sum_{n=1}^{\infty} \frac{1}{n^5} \le \frac{1}{4}+$ first term of the series $= \frac{1}{4} + 1$

In fact, by extending our work on $\sum_{n=1}^{\infty} \frac{1}{n}$ and $\sum_{n=1}^{\infty} \frac{1}{n^5}$ we can show that

$\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges if $p > 1$ and diverges if $p \le 1$.

9.1, 9.2 and 9.3 Do the following converge or diverge? Why? For series, choose from geometric series, terms not going to 0, linearity, or integral test, and show work as directed, including any assumptions.

$\sum_{n=1}^{\infty} e^n$

Tests that do not apply

Careful–the integral test does not apply. The terms are positive but not decreasing, so we can't use that test.

This is not a series made up of a linear combination of series so linearity does not apply

What can we use?

We can apply the terms not going to 0 test as

assumptions: $\lim_{n\to\infty} e^n$ goes to infinity since it is an exponential function which shows exponential growth.

So the series diverges by the terms not going to 0 test.

This is also a geometric series

assumptions: there is a constant fixed ratio from term to term x = e

Now $|x| = e > 1$ so the series diverges by geometric series test.

$s_n = \frac{1}{e^{2n}}$

This is a sequence, not a series. $\lim_{n\to\infty} \frac{1}{e^{2n}} = 0$ because this is the reciprocal of exponential growth, so as e^{2n} gets larger and larger, the reciprocal gets smaller and smaller towards 0.

None of the series tests apply here.

$\sum_{n=1}^{\infty} \frac{1}{e^{2n}}$

Tests that do not apply

The terms do go to 0 because the numerator stays fixed while the denominator gets larger by exponential growth, so that test is inconclusive and we must use another test.

This is not a series made up of a linear combination of series so linearity does not apply

What can we use?

This is a geometric series
assumptions: there is a constant fixed ratio from term to term $x = \frac{1}{e^2}$
The series converges because $|x| = \frac{1}{e^2} < 1$. The first term is $a = \frac{1}{e^2}$ so the series converges to $\frac{\frac{1}{e^2}}{1-\frac{1}{e^2}}$

Integral test also applies.
assumptions: The terms are decreasing and positive and this is a known integral,
so the series behaves the same as the integral $\int_1^\infty e^{-2x}\,dx = \lim_{b\to\infty}\int_1^b e^{-2x}\,dx = \lim_{b\to\infty}\frac{e^{-2x}}{-2}\Big|_1^b = \lim_{b\to\infty}\frac{e^{-2b}}{-2} - \frac{e^{-2\cdot 1}}{-2} = 0 + \frac{e^{-2}}{2}$
So the series converges since the integral does. We have the bound:
$\frac{e^{-2}}{2} \le \sum_{n=1}^{\infty}\frac{1}{e^{2n}} \le \frac{e^{-2}}{2} +$ first term $= \frac{e^{-2}}{2} + \frac{1}{e^2}$

Notice that the geometric series gives the exact convergence, while the integral test provides bounds.

$$\sum_{n=0}^{\infty}\frac{1}{1+n^2}$$

Tests that do not apply

The terms do go to 0 as the numerator is fixed while the denominator gets larger, so that test is inconclusive and we must use another test.

This is not a series made up of a linear combination of series so linearity does not apply (careful—don't have an algebra misstep as we can't break this up into 2 fractions when the sum is in the denominator).

This is not a geometric series. There is no fixed ratio from term to term, so we must use another test. It is not expressed as $\sum ax^n$ where x is a constant ratio between successive terms. To see this, you should get used to the forms that geometric series take. You can also compare successive terms. $\sum_{n=0}^{\infty}\frac{1}{1+n^2} = 1 + \frac{1}{2} + \frac{1}{5} + \ldots$ to notice that $\frac{\frac{1}{2}}{1} \neq \frac{\frac{1}{5}}{\frac{1}{2}}$.

This is not a series made up of a linear combination of series so linearity does not apply (careful—don't have an algebra misstep as we can't break this up into 2 fractions when the sum is in the denominator).

What can we use?

assumptions: the terms in the series are decreasing and positive and this is a known integral,
so the series behaves the same as the integral $\int_0^\infty \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\int_0^b \frac{1}{1+x^2}\,dx = \lim_{b\to\infty}\arctan x\Big|_0^b = \lim_{b\to\infty}\arctan b - \arctan 0 = \frac{\pi}{2} - 0$
So the series converges since the integral does. We have the bound:
$\frac{\pi}{2} \le \sum_{n=0}^{\infty}\frac{1}{1+n^2} \le \frac{\pi}{2} +$ first term $= \frac{\pi}{2} + 1.$

$$\sum_{n=2}^{\infty}\frac{1}{n(\ln n)^4}$$

Tests that do not apply

The terms do go to 0, so that test is inconclusive and we must use another test.

This is not a geometric series. There is no fixed ratio from term to term, so we must use another test. It is not expressed as $\sum ax^n$ where x is a constant ratio between successive terms. To see this, you should get used to the forms that geometric series take. You can also compare successive terms. $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4} = \frac{1}{2(\ln 2)^4} + \frac{1}{3(\ln 3)^4} + \frac{1}{4(\ln 4)^4} + \dots$ to notice that $\frac{\frac{1}{3(\ln 3)^4}}{\frac{1}{2(\ln 2)^4}} \neq \frac{\frac{1}{4(\ln 4)^4}}{\frac{1}{3(\ln 3)^4}}$.

This is not a series made up of a linear combination of series so linearity does not apply (careful—don't have an algebra misstep as we can't break this up into 2 fractions when the sum is in the denominator).

What can we use?

assumptions: the terms in the series are decreasing and positive, and this is a known integral, so the series behaves the same as the integral $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^4}\,dx = \lim_{b\to\infty} \int_2^{b} \frac{1}{x(\ln x)^4}\,dx$

Let $w = \ln x$ for substitution. Then $dw = \frac{1}{x}dx$ which is available in the integral:

$$\lim_{b\to\infty} \int_{x=2}^{x=b} \frac{1}{w^4}\,dw = \lim_{b\to\infty} \frac{w^{-3}}{-3}\Bigg|_{x=2}^{x=b} = \lim_{b\to\infty} \frac{\ln(x)^{-3}}{-3}\Bigg|_{x=2}^{x=b} = \lim_{b\to\infty} \frac{\ln(b)^{-3}}{-3} - \frac{\ln(2)^{-3}}{-3} = 0 + \frac{\ln(2)^{-3}}{3}$$

So the series converges since the integral does. We have the bound:

$$\frac{\ln(2)^{-3}}{3} \le \sum_{n=2}^{\infty} \frac{1}{n(\ln n)^4} \le \frac{\ln(2)^{-3}}{3} + \text{ first term } = \frac{\ln(2)^{-3}}{3} + \frac{1}{2(\ln 2)^4}$$