Applications of Differential Equations in Linear Algebra

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Course: Linear Algebra

1 Review of class Topics

1.1 Homogeneous Equations of Matrices

Solutions to the homogeneous equation $A\vec{x} = \vec{0}$ are pivotal to being able to compute differential equations.

Definition: A system of linear equations is said to be homogeneous if it can be written in the form $A\vec{x} = \vec{0}$, where A is an m x n matrix and 0 is a vector in $\mathbb{R}^m$. Such a system always has at least one solution, namely, $\vec{x} = \vec{0}$. This zero solution is usually called the trivial solution. When there is a solution that is not the zero vector and still satisfies the equation $A\vec{x} = \vec{0}$ it is considered the non-trivial solution.

The non-trivial solutions to a homogeneous can be found by augmenting A with zeros and employing **Gaussian Elimination**. For this project we were interested in finding the non-trivial solution to homogeneous equations. Homogeneous equations have nontrivial solutions when the Gaussian-reduced, augmented matrix has free variables.

Example:

$$\begin{pmatrix}1 & 2 & 3\\ 4 & 5 & 6\\ 7 & 8 & 9\end{pmatrix} \xrightarrow[r_3' = -7r_1 + r_2]{r_2' = -4r_1 + r_2} \begin{pmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 0 & -6 & -12\end{pmatrix} \quad r_3' = -2r_2 + r_3 \rightarrow \begin{pmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 0 & 0 & 0\end{pmatrix}$$

The matrix above then reduces by **Gaussian Elimination** to:

$$\begin{pmatrix}1 & 2 & 3\\ 0 & -3 & -6\\ 0 & 0 & 0\end{pmatrix}$$

The reduced form of the matrix tells us that there is a free variable, x_3, which would give us a **non-trivial solution**. We know that the Matrix above will have a free variable because it only has pivots in column one and column two with the third column having no pivot. Since this system has a free variable we know that it will have a non-trivial solution according to p. 43 of the text, which states: *The* ***homogeneous equation*** $A\vec{x} = \vec{0}$ *has a non-trivial solution if and only if the equation has at least one* ***free variable***

1.2 Determinants and Theorem 8

Definition: A determinant is an operation performed on a matrix. It is a measure of certain properties of the entries of a matrix expressed as a single number. The general form of the $det(A)$ operation is the **co-factor expansion**. From p. 165 of our text: "For $n \geq 2$ the **determinant** of a $n \times n$ matrix $A = [a_{ij}]$ is the sum of n terms of the form $\pm a_{1j} det(A_{1j})$, with plus and minus signs alternating, were the entries $a_{11}, a_{12}, ..., a_{1n}$ are from the first row of A...$[det(A)] = \sum_{j=1}^{n} (-1)^{1+j}\, a_{1j}\, det(A_{1j})$"

The general form is referred to a the **co-factor expansion**. For matrices of size $n < 4$, there are formulas which are a little simpler to apply. These are specific to the size of the matrix.

Formulas for $det(A_{2\times 2})$ and $det(A_{3\times 3})$:

$$det(A_{2\times 2}) = \begin{vmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{vmatrix} = a_{11} \cdot a_{22} - a_{21} \cdot a_{12}$$

$$det(A_{3\times 3}) = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{vmatrix} =$$

$$(a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32}) - (a_{13} \cdot a_{22} \cdot a_{31} + a_{11} \cdot a_{23} \cdot a_{32} + a_{12} \cdot a_{21} \cdot a_{33})$$

Cofactor Expansion Example:

$$det(A) = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 3 \end{vmatrix} = (-1)^{1+1} \cdot 1 \cdot \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{vmatrix}$$

Now we have reduced $det(A_{4\times 4})$ to $C \cdot det(A'_{3\times 3})$ where $C = (-1)^2 \cdot 1 = 1$, which we have a formula for.

$$1 \cdot det(A') = \begin{vmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{vmatrix} =$$

$$(a_{11} \cdot a_{22} \cdot a_{33} + a_{12} \cdot a_{23} \cdot a_{31} + a_{13} \cdot a_{21} \cdot a_{32}) - (a_{13} \cdot a_{22} \cdot a_{31} + a_{11} \cdot a_{23} \cdot a_{32} + a_{12} \cdot a_{21} \cdot a_{33})$$

Notice each multiplicative term in this expression contains a zero except for the first (the main diagonal), substituting our entries into this formula reduces to:

$$2 \cdot 1 \cdot 3 = 6$$

we can check our answer using Theorem 2 from p. 167 of the course textbook "If A is a triangular matrix [zeroes above or below the diagonal], then $det(A)$ is the product of the entries on the main diagonal of A." Since our original matrix has entries *only* on the diagonal we can be doubly sure it is triangular, so $det(A) = 1 \cdot 2 \cdot 1 \cdot 3 = 6$.

Whether a determinant is non-zero for example tells us a lot about a matrix, for example **a matrix with non-zero determinant is invertible**. This is a very important property because it allows is to tap into a set of properties fundamental to linear algebra.

Theorem 8: (excerpt from course textbook p. 112)
"Let a be A be a square $n \times n$ matrix. Then the following statements are equivalent. That is, for a given A, the statements are either all true of all false".

a. A is an invertable matrix
c. A has n pivot positions
d. The equation $A\vec{x} = \vec{0}$ has only the trivial solution
e. The columns of A form a linearly independent set.
h. The columns of A span $\mathbb{R}^n$.

Only the most pertinent elements of this theorem have been quoted here. This theorem is not only useful for the information if provides from being true it also tells us a substantial amount if its false. The relation between invertibility, span, and the non-trivial solutions of the homogeneous equation are essential to the computation or eigenvectors and eigenvalues.

1.3 Eigenvectors and Eigenvalues

Definition: An **eigenvector** of an n x n matrix A is a nonzero vector x such that $A\vec{x} = \lambda\vec{x}$ for some scalar λ. A scalar λ is called an **eigenvalue** of A if there is a nontrivial solution x of $A\vec{x} = \lambda\vec{x}$; such that $\vec{x}$ is called and eigenvector corresponding to λ.

Example: To find an **eigenvalue** of an $n \times n$ matrix is as follows:

$$A = \begin{pmatrix} 2 & 7 \\ -1 & -6 \end{pmatrix}, \lambda \times I = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}, A - \lambda I = \begin{pmatrix} 2-\lambda & 7 \\ -1 & -6-\lambda \end{pmatrix}$$

To find the non-complex Eigenvalues of the matrix A, want non-trivial solutions, we know from theorem 8 that if a matrix is not invertible if the determinant of a matrix is 0 (a.) and if so then the homogeneous equation $A\vec{x} = 0$ doesn't have only the trivial solution (d.):

$$det(A - \lambda I) = 0 = (2-\lambda)(-6-\lambda) - (-1 \cdot 7) = \lambda^2 + 4\lambda - 5 = (\lambda + 5)(\lambda - 1)$$

This looks suspiciously like a factored quadratic from precalculus, and we can interpret it the same way. The real eigenvalue of this equation is -5 and 1.

To find the **eigenvector** we can use the eigenvalue we found from the previous matrix, $\lambda = -5$ and $\lambda = 1$. Let $v = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ Then $(A = -5\lambda)\vec{v} = 0$ gives us:

$$\begin{pmatrix} 2+5 & 7 \\ -1 & -6+5 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

Then we get the equations:

$$\begin{pmatrix} 7v_1 & 7v_2 \\ -v_1 & -v_2 \end{pmatrix} \quad r_2' = (1/7)r_1 + r_2 \rightarrow \begin{pmatrix} 7v_1 & 7v_2 \\ 0v_1 & 0v_2 \end{pmatrix}$$

It is clear that we have one free variable(v_2) for the eigenvalue -5. Solving this system produces $v_1 = -v_2$ which makes our eigenvector at $\lambda = -5$ is $\begin{pmatrix} -1 \\ 1 \end{pmatrix}$. Computing for the eigenvalue $\lambda = 1$, using the same steps we used to solve for $\lambda = -5$, we get the eigenvector $\begin{pmatrix} -7 \\ 1 \end{pmatrix}$. Our eignevectors are linearly independent and span all of the $\mathbb{R}^2$. Note that all linear combinations $C_1 \cdot \begin{pmatrix} -1 \\ 1 \end{pmatrix} + C_2 \cdot \begin{pmatrix} -7 \\ 1 \end{pmatrix}$ are valid solutions.

In differential equations, both complex and non-complex solutions are allowed, we will see those applications in the following section.

2 Systems of Differential Equations

2.1 Definitions

Differential Equations: functions involving derivatives.

Ordinary Differential Equations: differential equations which do not have partial derivatives are referred to as **ordinary** differential equations.

Linear Differential Equations: differential equations which take the form $y' + Py = Q$. In other words, equations without exponents above 1 which contain derivatives.

Order: the order of a differential equation is determined by order of the derivatives in that equation. For example, an equation containing second derivatives is second order, those containing only first derivatives are first order

System of Differential Equations: Any system of equations whose elements contain derivatives.

Note: Most examples and concepts will be related to **Systems of Linear First Order Differential Equations.**

2.2 Fundamental Set of Solutions

We have a system of linear differential equations:

$$
\begin{aligned}
x_1' &= a_{11}x_1 + \cdots + a_{1n}x_n \\
x_2' &= a_{21}x_1 + \cdots + a_{2n}x_n \\
&\vdots \\
x_n' &= a_{n1}x_1 + \cdots + a_{nn}x_n
\end{aligned}
$$

These equations look very similar to the matrix equation $\vec{\mathbf{x}}'(t) = \mathbf{A}\vec{\mathbf{x}}(t)$

In fact we could treat these systems just like any other system of linear equations. We can treat the terms a_n as the coefficients of a matrix A and solve like any of the other systems of a $A \cdot \vec{x} = \vec{x}\,'$, remembering that $\vec{x}$ is related to $\vec{x}_n'$ not just by this equation, but also by differentiation (more about this in the coming sections). Such a solution is called the **fundamental set of solutions** for this system.

Example:
Let our system be:

$$\begin{aligned} x_1' &= 4x_1 & \longrightarrow x_1 &= C_1e^{4t} \\ x_2' &= -x_2 & \longrightarrow x_2 &= C_2e^{-t} \\ x_3' &= 3x_3 & \longrightarrow x_3 &= C_3e^{3t} \end{aligned} \quad \longrightarrow \begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{pmatrix} = \begin{pmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{pmatrix}$$

Where x_n and x_n' are functions of t. To the left of the first arrow we see the derivatives of each numbered function and on the right are the integrated functions using $y' = ky \to y = Ce^{kt}$ from calculus. Remember that our use of x and y in this case implies a function and not a single variable. To the right of the second arrow is the Linear Algebra interpretation of this system. Because each x_n' corresponds to only the term a_nx_n this system is said to be **decoupled** and our matrix A becomes a **diagonal** matrix.

2.3 Initial Value Problem

In our example above, we have constructed a matrix equation which give us the fundamental set of solutions which form a basis for the set of all solutions, but what about a specific, unique solution? The **initial value problem** can be solved if the $\vec{x}_0$ is known.

Example: Suppose we have a known value for each $x_n(0)$:

$$x_1(0) = 2$$
$$x_2(0) = -2$$
$$x_3(0) = 1$$

Plugging this in to our matrix equation from 2.2, we have

$$\begin{pmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 16 \\ 2 \\ 3 \end{pmatrix}$$

Recall that our functions are all of the form Ce^t, so we know also that at $t = 0$ $e^{kt} = 1$ for all k, so we have $C_n \cdot 1$ for all x_n:

$$\begin{pmatrix} 4C_1(1) \\ -1C_2(1) \\ 3C_3(1) \end{pmatrix} = \begin{pmatrix} 16 \\ 2 \\ 3 \end{pmatrix}$$

solving our simple system gives:

$$C_1 = 4$$
$$C_2 = -2$$
$$C_3 = 1$$

2.4 Eigenvalues and Differential Equations

For the above example, our solution $\vec{x}\,' = A\vec{x}$ is more or less evident, there being no interaction between our functions, but what if our system looked more like looked more like this (example from [3])?

$$y_1' = 3y_1 + 2y_2$$
$$y_2' = 6y_1 - 1y_2$$

then our A matrix would be would be:

$$\begin{pmatrix} 3 & 2 \\ 6 & -1 \end{pmatrix}.$$

We'll employ eigenvalues to help us toward a solution. Using maple or by-hand to compute the eigenvalues, we have:

$$\lambda_1 = -3 \text{ and } \lambda_2 = 5$$

and eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -3 \end{pmatrix} \text{ and } \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$$

Since our A isn't diagonal, wouldn't it be cool if we could make diagonal, turns out, we can! First, we'll construct a matrix V composed of our two eigenvectors

$$V = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix}$$

and we'll also need it's inverse. The determinant of V using the 2×2 formula is $(1 \cdot 1) - (-3 \cdot 1) = 4$ so we know from theorem 8 (section 1.2) that a non-zero determinant implies invertability, so we can be confident that our matrix V does have an inverse,

$$V^{-1} = \begin{pmatrix} 1/4 & -1/4 \\ 3/4 & 1/4 \end{pmatrix}$$

Now we can apply some matrix multiplication to get

$$V^{-1}AV = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix}$$

To explain the underlying theory of this would be beyond the scope of this project. Let it suffice to say that we have constructed a new, simpler system that nonetheless behaves as the old one did. Since it is a different system we probably shouldn't call it y_1 and y_2 anymore, lets use w_1 and w_2.

$$\begin{pmatrix} w_1' \\ w_2' \end{pmatrix} = \begin{pmatrix} -3 & 0 \\ 0 & 5 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and so we have

$$w_1' = -3w_1 \text{ where } w_1 = C_1 e^{-3t}$$
$$w_2' = 5w_2 \text{ where } w_2 = Ce^{5t}$$

Plugging the $\vec{y} = P\vec{w}$ relationship in to our original system, we have

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -3 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \end{pmatrix}$$

and so finally our solution is

$$y_1 = w_1 + w_2 = C_1 e^{-3t} + C_2 e^{5t}$$
$$y_2 = -3w_1 + w_2 = -3C_1 e^{-3t} + C_2 e^{5t}$$

Arriving at our solution is this manner, we have used what is known as the superposition of two solutions. If we can be confident they will produce the same result, a solution in one system is as good as another.

3 Annotated Sources

1. Linear Algebra and it's Applications (Course Text)

David C. Lay, Fourth Edition
Use: Source material for review topics and application of Linear Algebra to Differential Equations.

2. Mathematical Methods in the Physical Sciences

Mary L. Boas, Third Edition
Use: Definitions and examples regarding Differential Equations.

3. Elementary Linear Algebra

Larson/Edwards/Falvo, Fith Edition
Use: Applications of Linear Algebra in Differential Equations, section 7.4.

Maple 18

Use: Employed to verify any results used in examples.

Pauls Online Notes

$http : //tutorial.math.lamar.edu/Classes/DE/LA_Eigen.aspx$
Use: Additional examples

Special thanks to:

Dr Vicky Klima for LaTeX formatting help and 'soln' command.

Dr Sarah Greenwald for consultation regarding LaTeX formatting and Linear Algebra concepts.

Andrew Zeidell for consultation on Differential Equations concepts and source recommendations.