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> with(LinearAlgebra):  with(plots):
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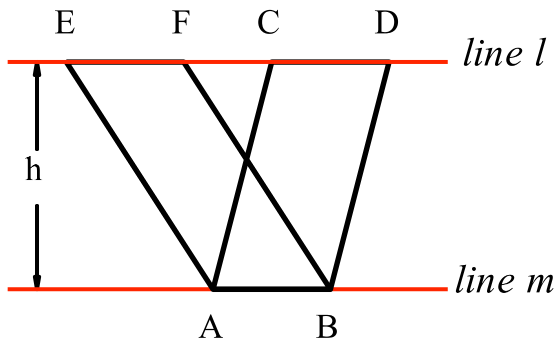
Geometry of Determinants and Row Operations

Adapted by Dr. Sarah from Visualizing Linear Algebra.

We begin by recalling a fact from geometry. In the figure below, lines *l* and *m* are parallel. Let *h* be the constant distance between these lines. In the figure we also have line segments AC parallel to BD and AE parallel to BF. These lines create two parallelograms: ACDB and AEFB.



The area of a parallelogram is the product of its base and height. Since both parallelograms have base AB and height *h*, they have equal areas. So if we think of the side AB of the parallelogram as fixed and the opposite side as sliding along the line *l*, we see that all such parallelograms have equal area. We record this fact:

- Parallelograms with a common base and with their side opposite the base lying on the same line have equal area.

We now apply this fact to determinants.

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> A := Matrix([[4,2],[2,6]]);
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$$A := \begin{bmatrix} 4 & 2 \\ 2 & 6 \end{bmatrix}$$

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> Determinant(A);
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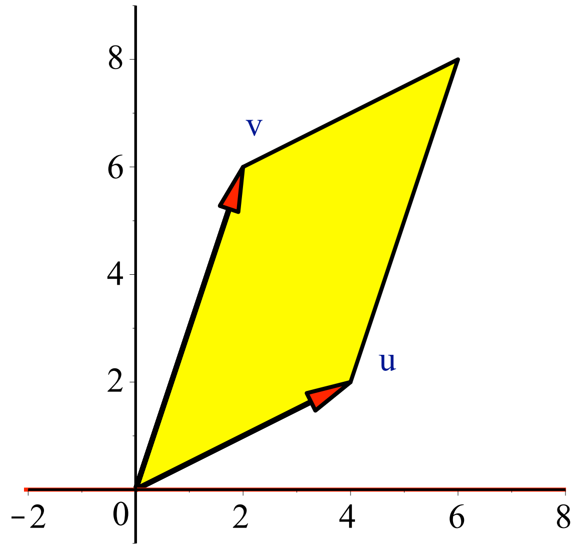
$$20$$

We form the unit span of the rows as vectors *u* and *v* (i.e., the parallelogram with the rows of the matrix, *u* and *v,* as two of its sides, or said another way, the portion of the plane t *u* + s *v* , where s and t range between 0 and 1):



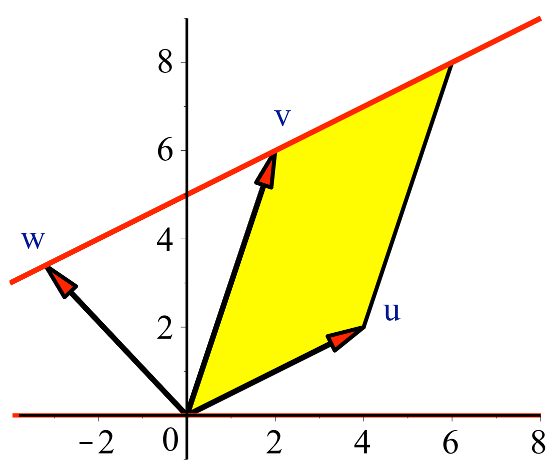
The area of this parallelogram is 20 square units, and it is also true that $det(A) = 20$. Why are these numbers the same? This example will answer that question.

The key to our understanding will be a geometric visualization of the familiar row reduction process. We will reduce the given 2 by 2 matrix A to a diagonal matrix and use the above geometric fact to show that the area of the unit span of the rows is unchanged. For a diagonal matrix, the area of the unit span of the rows is easy to compute.

Suppose we add some multiple k of row 1 (vector u) to row 2 (vector v). Then the resulting matrix B has the form:

$$B = \begin{bmatrix} u \\ w \end{bmatrix} = \begin{bmatrix} u \\ v + t\,u \end{bmatrix}$$

In particular, note that the new second row (vector w) will lie somewhere on the line $v + t\,u$ (the thick red line in the figure below, which is parallel to u and goes through the tip of v).



By our fact about areas of parallelograms, the area of the unit span of u and w equals the area of the unit span of u and v. Geometrically, we can say that our row operation slides one side of the unit span along a line parallel to the vector u, and therefore the area of the unit span is unchanged.

If in particular we take t = -1/2, then B is in echelon form.

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> B:=Matrix([[4,2],[0,5]]);
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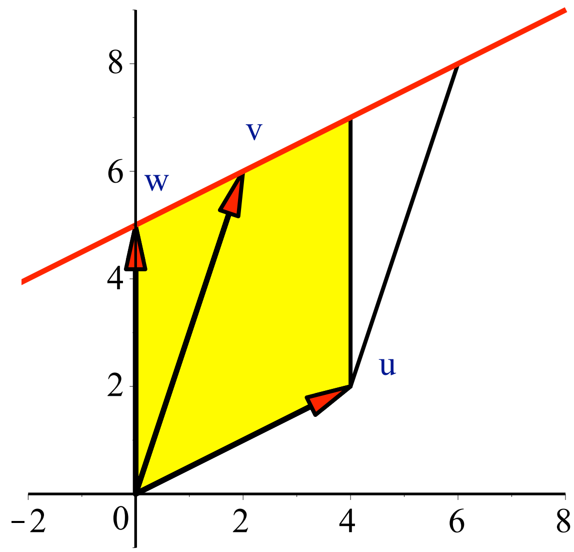
$$B := \begin{bmatrix} 4 & 2 \\ 0 & 5 \end{bmatrix}$$

The figure below shows the unit span of u and the particular vector $w = v - \frac{u}{2}$. Note that this vector w lies at the point where the line $v + t\,u$ intersects the y axis since its first component is 0.



To further row reduce matrix B, we next add -2/5 times row 2 to row 1.

```
> C:=Matrix([[4,0],[0,5]]);
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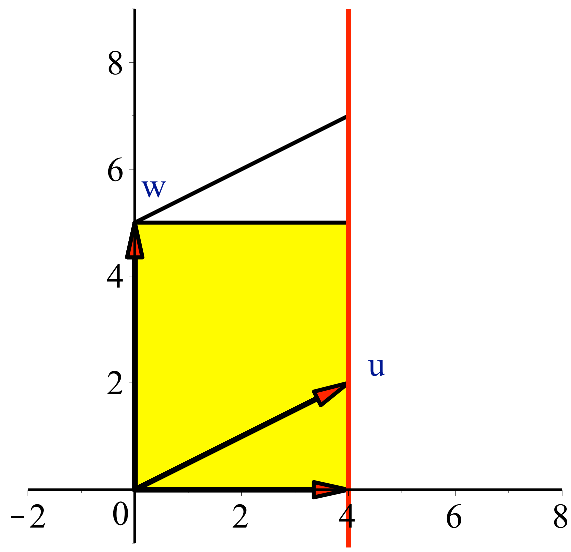
$$C := \begin{bmatrix} 4 & 0 \\ 0 & 5 \end{bmatrix}$$

Now we are adding a multiple of w (row 2) to u (row 1). So this time we slide one side of the parallelogram along a line parallel to vector w touching the tip of u until it touches the x axis, as shown below. Again the area of the parallelogram does not change, and the parallelogram becomes a rectangle. So the area of the rectangle is equal to the area of our original parallelogram.

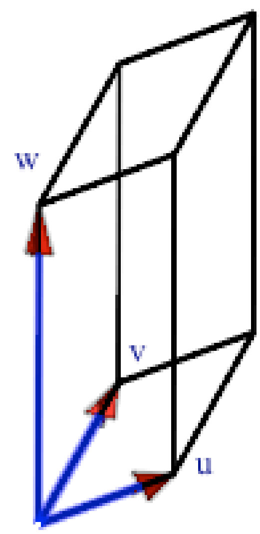


Finally, the area of the rectangle is (4)(5), which is also the determinant of F.
Let's summarize what we have seen. Whenever we add a multiple of one row to another, the area of the unit span of the resulting rows is unchanged. Furthermore, if we reduce the original matrix A until we have a diagonal matrix F, we will have transformed the unit span into a rectangle, whose area is easy to calculate and which clearly equals the determinant of F. But we also know that adding a multiple of one row to another has no effect on the value of the determinant. So we also have $det(A) = det(F)$. Therefore $det(A)$ equals the area of the unit span of its rows.

If we carry out this process on any 2 by 2 matrix, we might also need to use a row interchange. That would only introduce a sign change. So the general result is as follows:

- If A is a 2 by 2 matrix with row vectors u and v, then the area of the unit span of u and v equals the absolute value of $det(A)$.

In the event that u and v are parallel, then $det(A) = 0$ since A is singular. This is consistent with the unit span having zero area since geometrically it is a line segment. If A is a 3 by 3 matrix with row vectors u, v and w, then $|det(A)|$ (the absolute value of the determinant of A) is equal to the volume of the unit span of u,v and w (i.e., the parallelepiped with u, v and w as three of its sides), and row operations create a right rectangular prism or cuboid.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