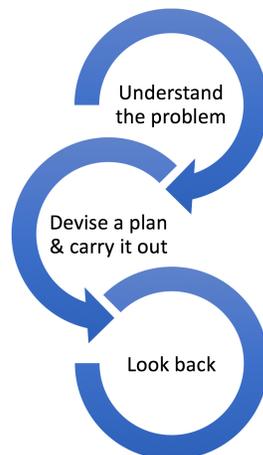


Problem Set 2

See the Guidelines and the Maple tips. **I will post on ASULearn answers to select questions I receive via messaging or in office hours. I am always happy to help!** You may work alone or in a group of up to 2 people and turn in one per group. The purpose of problem sets is to make connections and apply the material. Feel free to talk to me or each other if you are stuck, but be sure to acknowledge any sources outside me or your group, like “The insight for this solution came from a conversation with Joel.” **Be sure to annotate in your own group’s words.**



Mathematics, you see, is not a spectator sport. [George Polya, *How to Solve it*]

1. 1.3 #30—leave $\vec{v}_1, \dots, \vec{v}_k$ and k general, as this problem asks about generic vectors rather than ones with specific numbers, so do NOT use $\vec{v}_1, \dots, \vec{v}_4$ given in #29.

(a) First write out the general definition of \vec{v} as a linear combination of $\vec{v}_1, \dots, \vec{v}_k$.

(b) Next compare that to the general definition of \vec{v} as the center of gravity (as written in #29)

(c) The center of gravity definition already gives you a version very close to a linear combination, but you’ll need to do a teeny bit of algebra of vectors to modify that and solve for the weights c_i . Show reasoning.

2. Look at $L = \text{Span of } \left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} \right\} = \left\{ t \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix} \text{ where } t \text{ is real} \right\} = \left\{ \begin{bmatrix} t \\ 3t \\ -4t \end{bmatrix} \text{ where } t \text{ is real} \right\}$. Notice L is a line through the origin in \mathbb{R}^3 (1 free variable).

(a) Graph the vector in Maple (without the arrowhead) as follows:

```
with(LinearAlgebra): with(plots):
```

```
a:=spacecurve([1*t,3*t,-4*t],t=0..1,color=red, linestyle=solid):
```

```
display(a);
```

Notice the display command has a semi-colon but the other two have a colon to suppress the output.

(b) Find a vector \vec{w}_1 so that $\left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}, \vec{w}_1 \right\}$ spans a plane P_1 (and not just the $\begin{bmatrix} t \\ 3t \\ -4t \end{bmatrix}$ line) and describe how you found/selected it.

(c) Set up the following augmented matrix $\begin{bmatrix} 1 & * & b_1 \\ 3 & * & b_2 \\ -4 & * & b_3 \end{bmatrix}$, where the stars represent your vector \vec{w}_1 ,

and then explain how consistent solutions of this matrix relate to the concepts of span and/or linear combinations. This is not a computation—it is an annotation.

(d) Next use Gaussian in Maple on the matrix:

```
AugmentedPr2:=Matrix([Vector([1,3,-4]),Vector([your vector w1]),Vector([b1,b2,b3])]);  
GaussianElimination(AugmentedPr2);
```

(e) Write down the equation involving b_1, b_2 and b_3 that represents the plane that $\begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}$ and \vec{w}_1 lie in (set any inconsistencies equal to 0).

(f) Find a vector \vec{w}_2 not on the same line through the origin as \vec{w}_1 so that $\left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}, \vec{w}_2 \right\}$ also spans the same plane P_1 and describe how you chose \vec{w}_2 (Hint: the diagonal of the parallelogram would be a natural choice).

(g) In Maple, plot all three vectors on the same graph. You can use three different spacecurve commands each with a different color, like black, blue...

Be sure you have executed `with(plots):`

```
a:=spacecurve([1*t,3*t,-4*t],t=0..1,color=red, linestyle=solid);
```

```
b:=...
```

```
c:=...
```

```
display(a,b,c);
```

Or you can put all of the vectors together in one spacecurve command as the same color:

```
spacecurve({[1*t,3*t,-4*t],[vector2],[vector3]},t=0..1,color=red, linestyle=solid);
```

(h) Next, turn your graph “head on” to show that all three vectors lie on the same plane.

(i) Find a vector \vec{u} so that $\left\{ \begin{bmatrix} 1 \\ 3 \\ -4 \end{bmatrix}, \vec{u} \right\}$ spans a different plane through the origin. Describe how you found \vec{u} .

(j) In Maple, provide commands to plot all four vectors on the same graph. Turn the resulting graph “head on” to show that \vec{u} lies outside the original plane.

3. Concrete mix, which is used in jobs as varied as making sidewalks and building bridges, is composed of five main materials: cement, water, sand, gravel, and fly ash. By varying the percentages of these materials, mixes of concrete can be produced with differing characteristics. For example, the water-to-cement ratio affects the strength of the final mix, the sand-to-gravel ratio affects the “workability” of the mix, and the fly-ash-to-cement ratio affects the durability. Since different jobs require concrete with different characteristics, it is important to be able to produce custom mixes.

Assume you are the manager of a building supply company and plan to keep on hand three basic mixes of concrete from which you will formulate custom mixes for your customers. The basic mixes have the following characteristics:

	Super-Strong Type <i>S</i>	All-Purpose Type <i>A</i>	Long-Life Type <i>L</i>
Cement	19	18	12
Water	10	10	10
Sand	10	25	15
Gravel	16	5	15
Fly ash	5	2	8

Each measuring scoop of any mix weighs 60g, and the numbers in the table above give the breakdown by grams of the components of the mix. Custom mixes are made by combining the three basic mixes.

For example, a custom mix might have 3 scoops of Type S , 5 of Type A , and 2 of Type L . We can represent any mixture by a vector $\begin{bmatrix} c \\ w \\ s \\ g \\ f \end{bmatrix}$ in \mathbb{R}^5 representing the amounts of cement, water, sand, gravel, and fly ash in the final mix.

- Compute $3S + 5A + 2L$.
- Compare the workability of $3S + 5A + 2L$ to that of Types S , A and L —rank them in order of least to most workable (high sand to gravel ratio) and show work. Note that this is one example of a practical interpretation of vector coordinates.
- What does $\text{Span}\{S, A, L\} = \{aS + bA + cL \text{ where } a, b, \text{ and } c \text{ are real numbers}\}$ represent in this context—relate your answer to the word mixes? It is not a calculation. Instead, reinterpret the definition of span.
- A customer requests a custom mix with the following proportions of of cement, water, sand, gravel, and fly ash: 220:120:140:180:60. Is it possible to make using only S , A and L ? If so, find the weights (in scoops/parts of scoops) of the basic mixes (S , A , and L) needed to create this mix. If you choose to do this by-hand, don't forget to show the by-hand work, like row reduction. If you choose to use Maple, you can use commands like:

```
with(LinearAlgebra):
S := Vector([19,10,10,16,5]):
A := Vector([18,10,25,5,2]):
L := Vector([12,10,15,15,8]):
V := Vector([[220,120,140,180,60]]):
N:=Matrix([S,A,L,V]); ReducedRowEchelonForm(N);
```

- If there is a solution in Part (d), is the solution unique? Explain.

- Let $V = \begin{bmatrix} 220 \\ 120 \\ 140 \\ 180 \\ 60 \end{bmatrix}$. Explain why any linear combination of S , A , L and V can also be achieved by a combination of just S , A , and L , i.e. that V is redundant (think about substitution and using your answer from part (d)).

- What is the book's definition of linear independence using vectors?
- What is the definition of linear independence using a coefficient matrix A with the vectors as its columns ($A\vec{x} = \vec{0}$ has...)?
- Is $\{S, A, L, V\}$ linearly independent? Why or why not?

- Let $U = \begin{bmatrix} 12 \\ 12 \\ 12 \\ 12 \\ 12 \end{bmatrix}$. Show that $\{S, A, L, U\}$ is a linearly independent set of vectors by using the definition of l.i. If you choose to do this by hand, then show your reduction work and steps. If you choose to use Maple, and have already defined the vectors S , A and L as in the sample Maple commands in part (d), then you can use commands like:

```
U := Vector([12,12,12,12,12]):
```

```
zero:= Vector([0,0,0,0,0]):
```

```
SALUzero:=Matrix([S,A,L,U,zero]); ReducedRowEchelonForm(SALUzero);
```

Don't forget to annotate.

- (k) What practical advantage does $\{S, A, L, U\}$ being linearly independent have? For example, think about the mixes you can make using 4 linearly independent vectors compared to if you had 4 vectors that were not l.i. (i.e. dependent vectors). Say something beneficial about l.i. in this context.
- (l) Compute the mix $S + A + L + U$.
- (m) Next, modify the first entry of the vector $S + A + L + U$ to define a fifth basic mix W to add so that any custom mixture can be expressed as a linear combination of the set of mixes $\{S, A, L, U, W\}$. To show your mix works, set up the augmented system $\{S, A, L, U, W, \text{generic vector}\}$ and use GaussianElimination in Maple to show that your vectors span \mathbb{R}^5 . Once you define W in Maple using a Vector command, you can execute:

```
W := ...
```

```
SALUWspan:=Matrix([S,A,L,U,W,Vector([b1,b2,b3,b4,b5])]);
```

```
GaussianElimination(SALUWspan);
```

- (n) In real-life, there will still be mixes that cannot be physically produced from this set of five basic mixes. Give an example of a custom mix where at least one weight (of S, A, L, U, W) is negative and in your annotations address why we can't use negative weights in mixing cement in real-life.

A Review of Various Maple Commands:

```
with(LinearAlgebra): with(plots):
```

```
A:=Matrix([[ -1,2,1,-1],[2,4,-7,-8],[4,7,-3,3]]);
```

```
ReducedRowEchelonForm(A);
```

```
GaussianElimination(A); (only for matrices with generic entries like  $k$  or  $a, b, c$ )
```

```
Vector([1,2,3]);
```

```
A.B;
```

```
A+B;
```

```
B-A;
```

```
3*A;
```

```
A^3;
```

```
evalf(M); (decimal approximation of M)
```

```
spacecurve({[4*t,7*t,3*t],[-1*t,2*t,6*t]},t=0..1, color=red, linestyle=solid); plot vectors as line segments in  $\mathbb{R}^3$  (i.e. the columns of matrices) to show whether the columns are in the same plane, etc.
```

```
implicitplot({2*x+4*y-2,5*x-3*y-1}, x=-1..1, y=-1..1);
```

```
implicitplot3d({x+2*y+3*z-3,2*x-y-4*z-1,x+y+z-2},x=-4..4,y=-4..4,z=-4..4); plot equations of planes in  $\mathbb{R}^3$  (rows of augmented matrices) to look at the geometry of the intersection of the rows (i.e. 3 planes intersect in a point, a line, a plane, or no common points)
```

```
display(a,b,c); if plots are set to variables then this will display them on the same plot
```