

## Problem Set 3

See the Guidelines and the Maple tips. **I will post on ASULearn answers to select questions I receive via messaging or in office hours. I am always happy to help!** You may work alone or in a group of up to 2 people and turn in one per group. The purpose of problem sets is to make connections and apply the material. Feel free to talk to me or each other if you are stuck, but be sure to acknowledge any sources outside me or your group, like “The insight for this solution came from a conversation with Joel.” **Be sure to annotate in your own group’s words.**



*Mathematics, you see, is not a spectator sport.* [George Polya, *How to Solve it*]

1. 2.1 # 18 with additional instructions:

- Is the third column of  $AB$  zero or not? Why? Be sure to leave  $A$  and  $B$  general—you don’t know what size they are, nor what the entries are, aside from what you are given in the problem. However, you should use the column definition of matrix multiplication rather than the dot product definition—of  $A$  times each column of  $B$ —and reason from there.
- What is the definition of linear independence using a coefficient matrix  $AB$  with the vectors as its columns ( $AB\vec{x} = \vec{0}$  has...)?
- What do the prior parts tell you about whether the columns of  $AB$  are l.i. or not or whether there is no way to tell? Consider  $\vec{x}$  with a nonzero entry in the 3rd spot but zeros everywhere else. Explain in your annotation.

2. 2.2 #12—show all linear algebra steps, annotate your reasoning, and use associativity in your reasoning. Be careful about what side you multiply things by—be sure it is the correct side that makes sense in the context (because multiplication is not commutative).

3. Adapted from 2.3 #43

- After the usual first commands of `with(LinearAlgebra): with(plots):`, enter the matrix from 2.3 #10 into Maple as follows—define it to be  
`A:=Matrix([[5,3,1,7,9],[6,4,2,8,-8],[7,5,3,10,9],[9,6,4,-9,-5],[8,5,2,11,4]]);`
- Compute `ConditionNumber(A);` in Maple.
- Either in text comments or by-hand, write the condition number in scientific notation, and specify what is  $k$ , the order, i.e. the power of 10 in scientific notation.

(d) Solve  $A\vec{x} = \begin{bmatrix} 5 \\ 6 \\ 7 \\ 9 \\ 8 \end{bmatrix}$  via the following Maple command:

`MatrixInverse(A).Vector([5,6,7,9,8]);`

- (e) Explain why the Maple output makes sense in this context—compare  $\begin{bmatrix} 5 \\ 6 \\ 7 \\ 9 \\ 8 \end{bmatrix}$  to the columns of  $A$

and relate it to the linear algebra of it as a linear combination of the columns with your answer in part (d) as the weights.

- (f) Execute the following in Maple:

`MatrixInverse(A).Vector([5.1,6.1,7.1,9.1,8.1]);`

`MatrixInverse(A).Vector([5.01,6.01,7.01,9.01,8.01]);`

`MatrixInverse(A).Vector([5.001,6.001,7.001,9.001,8.001]);`

`MatrixInverse(A).Vector([5.0001,6.0001,7.0001,9.0001,8.0001]);`

`MatrixInverse(A).Vector([5.00001,6.00001,7.00001,9.00001,8.00001]);`

`MatrixInverse(A).Vector([5.000001,6.000001,7.000001,9.000001,8.000001]);`

- (g) Notice that when we use decimals in the vector  $\vec{b}$  (like from `Vector([5.1,6.1,7.1,9.1,8.1]);`), then the vector solution  $\vec{x}$  of  $A\vec{x} = \vec{b}$  (via  $A^{-1}\vec{b}$ ) changes quite a lot from part (d) where we didn't use decimals. This is obviously not very good for fields like engineering where all measurements, constants and inputs are approximate.

How many significant digits do we need (count 5.01 as 3 significant digits) in  $\vec{b}$  in order to have `x=MatrixInverse(A).b` in Maple give us an equivalent solution, rounded to one decimal place, to the output in part (d). (Hint: 0.0727999... is not equivalent to 0 when rounded to one decimal place, as it rounds to .1, but .0072... is equivalent to 0 when rounded to one decimal place—once you have the one that works, how many significant digits does the corresponding  $\vec{b}$  have?).

- (h) Use  $r$  as your previous answer in (g), and  $k$  from part (c) in order to plug into  $r - k$  as in the instructions above 2.3 # 42 and # 43 in the book.
- (i) Does the book's assertions about accuracy (in those instructions above 42 and 43) match what you found here?

4. Assume that you intercept a number of items, as follows:

- The following text sent from person 1: *A system of inconsistent equations walks into a doctors office*
- A string of coded numbers as a reply: -2, -5, -12, -13, 9, 2, -3, -3, 25, 10, 6, 0, 15, 6, 13, 4, 8, -6, 38, 19
- The beginning of the decoded message: *Chan*
- The fact that a 2x2 decoding matrix was used in the Hill Cipher

What does the doctor prescribe? We'll investigate whether the rest of the message can be decoded as follows:

- (a) Set up two matrix vector equations for the decoding matrix  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$  using the intercepted part of the message, either by-hand or in Maple and annotate as below:

Define the decoding matrix using generic variables  $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$

$$\begin{bmatrix} -2 \\ -5 \end{bmatrix} \text{ decodes to the vector corresponding to "ch" via } \begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} -2 \\ -5 \end{bmatrix} = \begin{bmatrix} 3 \\ 8 \end{bmatrix}$$

Now you set up the next part of the string  $\begin{bmatrix} -12 \\ -13 \end{bmatrix}$  decoding to the vector corresponding to “an”—you create the vector for “an” as well as the matrix vector equation and annotate the linear algebra like I did just above, but for this part of the message.

(b) Multiply the systems by-hand or in Maple. In Maple you can use:

```
with(LinearAlgebra): with(plots):
DecodingMatrix:=Matrix([[a, b],[c,d]]);
DecodingMatrix.Vector([-2,-5]) = Vector([3,8]);
DecodingMatrix.Vector([-12,-13]) = ; where you put your vector in
```

(c) From setting equal each corresponding entry in part (b), write down the 4 equations in the 4 unknowns  $a, b, c, d$ .

(d) Solve this system (it’s linear—you can solve it using a variety of methods, like a  $4 \times 5$  augmented matrix with columns  $abcd$  and an equals column, so your first row would be  $[-2,-5,0,0,3]$ ) and `ReducedRowEchelonForm` to see whether you have 0, 1 or infinite solutions for  $a, b, c$ , and  $d$  (a review of those commands are at the end of this document).

(e) If you have solutions, put them into `DecodingMatrix:=Matrix([[a, b],[c,d]])`; (careful about which numbers are in which spots) and use this to decode (by-hand or in Maple). If you are using Maple, then first define your `DecodingMatrix`.

```
DecodingMatrix:= ; you define
CodedMessage:= Matrix([[-2,-12,9,-3,25,6,15,13,8,38],[-5,-13,2,-3,10,0,6,4,-6,19]]);
DecodingMatrix.CodedMessage;
```

You must define the `DecodingMatrix` correctly using part (d). Notice also that the coded string has gone in as the column vectors of the `CodedMessage`.

(f) Then translate back to letters by reading down the columns to explain what the doctor prescribed. If there are no solutions to the cipher, then explain why the system is inconsistent.

A Review of Various Maple Commands:

```
with(LinearAlgebra): with(plots):
A:=Matrix([[-1,2,1,-1],[2,4,-7,-8],[4,7,-3,3]]);
ReducedRowEchelonForm(A);
```

`GaussianElimination(A)`; (only for matrices with generic entries like  $k$  or  $a, b, c$ )

`ConditionNumber(A)`; (only for square matrices)

`B:=MatrixInverse(A)`; (only for square matrices)

`Transpose(A)`;

`Vector([1,2,3])`;

`A.B`;

`A+B`;

`B-A`;

`3*A`;

`A^3`;

`evalf(M)`; (decimal approximation of  $M$ )

`spacecurve({[4*t,7*t,3*t],[-1*t,2*t,6*t]},t=0..1, color=red, linestyle=solid)`; plot vectors as line segments in  $\mathbb{R}^3$  (i.e. the columns of matrices) to show whether the columns are in the same plane, etc.

`implicitplot({2*x+4*y-2,5*x-3*y-1}, x=-1..1, y=-1..1)`;

`implicitplot3d({x+2*y+3*z-3,2*x-y-4*z-1,x+y+z-2},x=-4..4,y=-4..4,z=-4..4)`; plot equations of planes in  $\mathbb{R}^3$  (rows of augmented matrices) to look at the geometry of the intersection of the rows (i.e. 3 planes intersect in a point, a line, a plane, or no common points)

`display(a,b,c)`; if plots are set to variables then this will display them on the same plot