

Problem Set 4

See the Guidelines and the Maple tips. **I will post on ASULearn answers to select questions I receive via messaging or in office hours. I am always happy to help!** You may work alone or in a group of up to 2 people and turn in one per group. The purpose of problem sets is to make connections and apply the material. Feel free to talk to me or each other if you are stuck, but be sure to acknowledge any sources outside me or your group, like “The insight for this solution came from a conversation with Joel.” **Be sure to annotate in your own group’s words.**



Mathematics, you see, is not a spectator sport. [George Polya, *How to Solve it*]

1. 3.1 # 46

After the usual first commands of `with(LinearAlgebra): with(plots):` entering `A:=RandomMatrix(4);` will give you a 4×4 matrix with integer entries and `Determinant(A);` will compute the determinant of the matrix A in Maple, which is part of what you are directed to do by the book. Complete 3.1 # 46 using these types of commands as well as `MatrixInverse(A);`. Don't forget to annotate.

2. 2.8 #38 with the following instructions:

- What is our book's definition (or the glossary) of the nullspace of a matrix A , or $\text{Null } A$?
- For the matrix listed in the book problem, what is the augmented matrix for the homogeneous equation corresponding to $\text{Null } A$?
- Use the `ReducedRowEchelonForm` command in Maple on the augmented matrix in part (b) (after you define it as a matrix in Maple—see below for a review of such commands)
- Continue with the reduced matrix in part (c) to show by-hand work to solve for $\text{Null } A$.
- What is the geometry of $\text{Null } A$? Choose one from [point, line, plane, entire space] and briefly explain why in your annotations.
- What is our book's definition (or the glossary) of the column space of a matrix A , or $\text{Col } A$?
- Solve for $\text{Col } A$ as follows: reduce A , circle the pivots and provide the pivot columns of A (not reduced A) as the basis for the $\text{Col } A$. Note that linear combinations of these basis vectors span $\text{Col } A$.
- To look for another representation of $\text{Col } A$, set up and solve the augmented matrix for \vec{x} in the system $A\vec{x} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix}$ by augmenting A with a generic vector and using the `GaussianElimination` command.
- Are there any inconsistent parts (like $[0 \ 0 \ 0 \ 0 \ 0]$ combination of bs) to set equal to 0 and provide an algebraic representation of $\text{Col } A$ in terms of b_1, b_2, b_3 and/or b_4 ?

- (j) What is the geometry of Col A ? Choose one from [point, line, plane, entire space] and briefly explain why in your annotations.
3. 5.6 #10 with the following directions:
- What is our book's definition (or the glossary) of an eigenvalue?
 - What is our book's definition (or the glossary) of an eigenvector?
 - By-hand or in Maple compute the eigenvalues and eigenvectors of the matrix given in the problem. If you are in Maple, don't forget to use fractions instead of decimals if you are using `Eigenvec-tors(A)`;
 - Briefly explain why the eigenvectors span all of \mathbb{R}^2 .
 - Write out the eigenvector decomposition for the system.
 - Do the populations grow, die off, stabilize, or exhibit some other behavior in the long run for most starting positions? Annotate how you can tell from the eigenvector decomposition.
 - What is the eventual ratio of the populations the system tends to in the long run for most starting positions? Annotate how you can tell from the eigenvector decomposition.
 - For most starting positions, what is the yearly rate (growth rate, die off rate, or stability rate) in the long run? Annotate how you can tell from the eigenvector decomposition.
 - Sketch a by-hand plot of the two eigenvectors.
 - Add to your plot to make it a trajectory plot by selecting a starting position in the 1st quadrant that is not on either eigenvector and demonstrating the long-term behavior.
4. Rotation matrices in \mathbb{R}^2 . Recall that the general rotation matrix which rotates vectors in the counter-clockwise direction by angle θ is given by `M:=Matrix([[cos(theta),-sin(theta)],[sin(theta),cos(theta)]])`;
- What is our book's (or the glossary) definition of a linear transformation on \mathbb{R}^n ?
 - Apply the `Eigenvalues(M)` command (Eigenvalues, not Eigenvectors here) in Maple or solve for the eigenvalues by-hand.
 - Notice that there are real eigenvalues for certain values of θ only. What are these values of θ ? Show work/reasoning. (Recall that I or i , $\sqrt{-1}$, does not exist as a real number and that $\cos(\theta) \leq 1$ always. You'll want to use this in your annotation.)
 - For each such value of θ that gives real-eigenvalues, what eigenvalues do they produce? Don't forget to look for other possible eigenvalues!
 - For each θ that gives a real eigenvalue, find a basis for the corresponding eigenspace (`Pi` is the correct way to express π in Maple. You can use commands like `Eigenvalues(Matrix([[cos(2*Pi),-sin(2*Pi)],[sin(2*Pi),cos(2*Pi)]])`); in Maple and reason from there, or reason from by-hand work.
 - Use only a geometric explanation to explain why most rotation matrices have no eigenvalues or eigenvectors and include how the rotation angle connects. Hint: how do rotations act on vectors physically—how does this relate to a line any vector was on for most θ —so they do not scale along any line through the origin to realign vectors?

A Review of Various Maple Commands:

`with(LinearAlgebra): with(plots):`

`A:=Matrix([[-1,2,1,-1],[2,4,-7,-8],[4,7,-3,3]]);`

`ReducedRowEchelonForm(A);`

`GaussianElimination(A);` (only for matrices with generic entries like k or a, b, c)

`ConditionNumber(A);` (only for square matrices)

`B:=MatrixInverse(A);` (only for square matrices)

`Transpose(A);`

`Eigenvalues(A);`

`Eigenvectors(A);`

`evalf(Eigenvectors(A));` decimal approximation

`Vector([1,2,3]);`

`A.B;`

`A+B;`

`B-A;`

`3*A;`

`A^3;`

`evalf(M);` (decimal approximation of M)

`spacecurve({[4*t,7*t,3*t],[-1*t,2*t,6*t]},t=0..1, color=red, linestyle=solid);` plot vectors as line segments in \mathbb{R}^3 (i.e. the columns of matrices) to show whether the columns are in the same plane, etc.

`implicitplot({2*x+4*y-2,5*x-3*y-1}, x=-1..1, y=-1..1);`

`implicitplot3d({x+2*y+3*z-3,2*x-y-4*z-1,x+y+z-2},x=-4..4,y=-4..4,z=-4..4);` plot equations of planes in \mathbb{R}^3 (rows of augmented matrices) to look at the geometry of the intersection of the rows (i.e. 3 planes intersect in a point, a line, a plane, or no common points)

`display(a,b,c);` if plots are set to variables then this will display them on the same plot