The History of Group Theory

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Group theory is now an accepted part of abstract algebra, a field that can stand alone in mathematics like analysis, geometry, or number theory. Yet, it did not always have this status: it took many years and small steps to gain its rightful place in the discipline of mathematics. Its origins can be traced to the 1700s, though earlier Greek mathematicians did pose problems that are made more understandable today using group theory. Its major thrust to achieve its modern status came in the mid 1800s when Arthur Cayley led the charge to study abstract structures solely for their beauty alone and not for their application to other branches of mathematics. Now, this theoretical topic is widely acknowledged to be a necessary and integral part of any mathematics education. This paper seeks to touch on a few of the main developments in the rise of group theory; it does not claim to detail an exhaustive list but only to give a cursory glance at this important record of events.

Symmetry was one of the biggest factors in the beginnings of group theory. As early as the 13^{th} century, the Moors were involved in creating mosaics filled with symmetry. The Alhambra, which is located in southern Spain, stands as a monument to their ingenious creation and clear perception of symmetry. In the 15^{th} century, Leonardo da Vinci also explored the realm of symmetry and determined all possible symmetry "groups" of planar objects (Class notes, 2003). Later, in the early 1700s, Muhammad al-Fullani al-Kishnawa evaluated the symmetries of a square to help in his study of magic squares (Greenwald, 2003). Also, Joseph Lagrange experimented in the late 1700s with permutations: how many different ways the elements of a set could be arranged. All of these concepts contributed to the idea of a group, yet the true beginnings of an actual theory began with Evariste Galois in the early 1800s. It was he who first coined the term *group*. After Niels Abel proved the insolubility of the quintic equation, Galois soon followed in his research using "substitution groups to determine the algebraic solvability of equations…" (Cajori, 1991, p. 352). So, along with symmetry, the search for solutions to algebraic equations led to the discovery of the mathematics of groups.

In the 1840s, group theory began to gain a greater following among mathematicians. William Hamilton invented the quaternions, a group that has many important applications in three and four dimensions. Augustin-Louis Cauchy published what is now known as Cauchy's Theorem. This theorem shows knowledge of the definition of order of both groups and elements of groups as well as the concept of a subgroup. Cauchy's Theorem states that each group whose order is divisible by a prime p must have at least one subgroup of order p. The foundations were being laid for a full-fledged theory to be developed. The importance mathematicians were giving the subject are evidenced by the fact that Joseph Serret began to teach a course in group theory in 1848 to his students in Paris.

The 1850s brought the advent of a formal group theory, which was put forth by Arthur Cayley. He gave a definition for groups (in which he gave credit to Galois for the foundations) that was similar to the modern definition, yet it lacked a few things. Apparently, Cayley assumed several facts about the set that he defined to be a group, facts that modern mathematicians do not take for granted; i.e., that the set is finite in order, that multiplication is associative, and that every element has an inverse (Katz, 1998).

Though he was a bit premature in the definition of a group, Cayley did contribute greatly to the field. He succeeded in making group theory an abstract study of its own: he could make up any set of elements and describe their relationship to each other within the set, and he was not necessarily concerned with the applications this individual set might have to other areas of mathematics or physics. This was a huge step for this mathematics; now it was worthy of being studied based on its own qualities. Other contributions of Cayley include his invention of Cayley tables to describe the interaction between elements of a group and his proof that the quaternions were a group of order eight under multiplication. This last point was significant, for only permutations had been considered as groups until Cayley demonstrated that both the quaternions and matrices were groups (O'Connor & Robertson, 2003). Cayley's service to the theory of groups is beyond compare.

What was now needed was the axiomatic definition for what properties defined a group. Two men delivered the needed definitions in the year 1882. Walter Dyck provided the basis for the axiomatic definition only leaving out the inverse and associative properties, though he did appear to allude to them (Katz, 1998). Later that year, Heinrich Weber gave the definition that very closely resembles that which is used today:

> A system *G* of *h* arbitrary elements $\square_1, \square_2, \ldots, \square_h$ is called a group of degree *h* if it satisfies the following conditions:
>
> I. By some rule which is designated as composition or multiplication, from any two elements of the same system one derives a new element of the same system. In symbols $\square_r \square_s = \square_t$.
>
> II. It is always true that $(\square_r \square_s)\square_t = \square_r(\square_s \square_t) = \square_r \square_s \square_t$.
>
> III. From $\square\square_r = \square\square_s$ or $\square_r \square = \square_s \square$ it follows that $\square_r = \square_s$. (Burton, 2003, p. 602)

Weber also defined a group to be Abelian (in honor of Niels Abel) if the multiplication was commutative. This was the last step to the creation of formal theory. Groups now had an axiomatic definition on which to lean as well as many important theorems and properties that had already been established.

Thus, the journey to an established theory of groups took over 500 years. The Moors provided the basis with their study of symmetry, and many years later Weber produced the crowning definition. Today, mathematicians continue to study groups of all types—infinite, finite, Abelian, simple, and others. The history is not complete; it has only begun.

Selected Bibliography

Burton, D.M. (2003). *The History of Mathematics: An Introduction* (5th ed.). New York: McGraw-Hill.

Cajori, F. (1991). *A History of Mathematics* (5th ed.). New York: Chelsea.

Class notes: History of Mathematics. Spring 2003.

Greenwald, S. The symmetries of a Square [Electronic Version]. Retrieved April 30, 2003 from http://www.cs.appstate.edu/~sjg/class/3010/square.html

Katz, V.J. (1998). *A History of Mathematics: An Introduction* (2nd ed.). New York: Addison-Wesley.

O'Connor, J.J. & E.F. Robertson. The development of group theory [Electronic version]. Retrieved April 16, 2003, from http://turnbull.mcs.st-nd.uk/~history/HistTopics/Development_group_theory.html