Worksheet on Hyperbolic Geometry Part 4

Dr. Sarah's MAT 3610: Introduction to Geometry

goals:

- IGS Exploration
  I can use Interactive Geometry Software (IGS) to discover relationships and demonstrate they seem to apply in a wide variety of examples.
- Proof Considerations
  I can write rigorous proofs in geometry, identify underlying assumptions, and understand limitations and applications.
- Geometric Perspectives
  I can compare and contrast multiple geometric perspectives.

Welcoming Environment: Keep it a safe place to express meaningful ideas and opinions. Actively listen to others and encourage everyone to participate. Part of the welcoming environment is to keep an open mind as you engage in our class activities, explore consensus and employ collective thinking across barriers. Maintain a professional tone, show respect and courtesy, and make your contributions matter.

1. **Building Community**: What are the preferred first names of those sitting near you? If you weren't able to be there write N/A or give reference to anyone you had help from.

Another Model of Hyperbolic Geometry

2. Search the web to find the measure of one interior angle of a flat octagon. What is it?

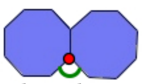
3. To understand why that is the interior angle, first subdivide the octagon into triangles that all emanate from the same one vertex at one corner of the octagon. Write a proof and identify assumptions—you should use that the sum of the angles in a flat triangle is 180°—and also create an accompanying sketch.

4. On an octagon we glue the side with a number on it with the side that has the same number on it. It is an exercise in visualization skills to see that the resulting figure is a 2-holed donut:

Images: Sascha Rogmann http://www.rogmann.org/math/tori/torus2en.html

To understand why Euclidean geometry does not apply to the 2-holed donut, we can look to see whether octagons will tile the plane or not. So we would like to know whether we can take a certain number of octagons (instead of angels and demons like Escher used in the distorted hyperbolic *Heaven and Hell* work...) and put them together around a vertex in order to form 360°. First, if we put two of them together, and want to understand how much angle they take up, we can double a single interior angle of a flat octagon—since they each take up the same amount of space. So double your angle from your response in #2:

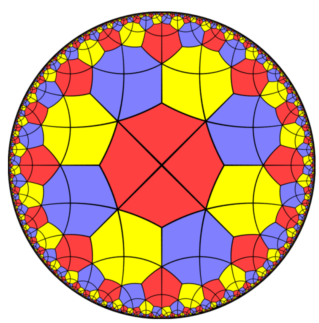
5. How much of 360° is left over at the red point when two octagons are placed side by side—the leftover angle is indicated by the green arc (360°− response from #4)?



6. What happens if we try and place three octagons together at a vertex? Could they fit into 360°?

7. We can create a 2-holed donut by using distorted octagons with 45° interior angles that fit together to tile hyperbolic space. Eight of these glue together like in Escher's work to form 360° at a vertex and so they tile the space ($45 \times 8 = 360$). Now we understand some of the issues that Escher faced, why his *Heaven and Hell* work looked like it did, and why these spaces are not flat—in hyperbolic geometry sides bow in to be able to fit more together. Sketch over the edges of one of the hyperbolic octagons from the middle image of hyperbolic octagon tilings and also put a * there:

crocheted hyperbolic octagon



CC-BY 2.5 by Claudio Rocchini

Circle Limit 4: Heaven and Hell by M.C. Escher, 1960

SAS in various geometries

8. Review the proof of proposition 4 in *Euclid's Elements* Book I via your notes from the congruence and similarity 1 interactive video. Where does the Euclidean proof first fail in taxicab geometry? Sketch two different taxicab triangles from the analytic geometry and metric perspectives interactive video and write down the first part of the proof that fails.

9. Where does the Euclidean proof first fail in spherical geometry? Sketch two different spherical triangles with the same SAS from the Euclidean and spherical perspectives interactive video and write down the first part of the proof that fails.

10. Is I-4 true for hyperbolic triangles? Show a hyperbolic sketch and write down each of the steps in the Euclidean proof and annotate whether each step works in hyperbolic geometry or not (like a checkmark).

11. **Help each other and PDF responses to ASULearn**: If you are finished with the worksheet before I bring us back together, first ensure that your entire group is finished too, and if not, help each other. If your entire group is finished, then split up and pull up chairs so that you can discuss your responses with other groups. Collate your handwritten responses, preferably on this handout, into one full size multipage PDF for submission in the ASULearn assignment. I recommend you turn it in sometime today, but you have until the next class.